THE INVARIANT THEORY OF THE INVERSION GROUP: GEOMETRY UPON A QUADRIC SURFACE*

BY

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INTRODUCTION.

The extended notion of invariant developed by KLEIN and LIE has become one of the fundamental concepts of mathematical thought: *to every group of transformations there corresponds a geometry, or theory of invariants, dealing with those properties of geometrical or analytical configurations, which are unaltered by the group.** Of the theories which are thus possible few have actually been developed, the most important, of course, being that based upon the total group of linear transformations, i. e., projective geometry or the ordinary theory of forms. The theory of other groups also possessing an algebraic invariant theory has recently received well-deserved consideration, the most prominent investigator being STUDY, who has applied himself to various important subgroups of the general projective group. A remarkable advance in the theory of such groups has been made by MAURER, who has proved that algebraic forms possess a complete system of concomitants not merely with respect to the total group of linear transformations (as had been shown by HILBERT), but also with respect to any subgroup, i. e., with respect to *any* linear group.†

The present paper is concerned with projective geometry upon a non-degenerate quadric surface, or, more specifically, with

(A) *the theory of the algebraic curves upon a proper quadric surface, with respect to those properties which are unaltered by the group of collineations transforming the quadric into itself.*

* KLEIN: *Erlanger Programme* (1872); *Höhere Geometrie* (1893).

† Oral communication to the writer by Professor HILBERT; MAURER'S paper in the Münchner Sitzungsberichte was at the time inaccessible. The method employed by MAURER is that developed by HILBERT: *Ueber die Theorie der algebraischen Formen*, Mathematische Annalen, vol. 36, pp. 473–534, 1890, the Ω process of p. 525 being generalized. Whether there are non-linear groups with algebraic invariant theories is still, so far as I know, an unsettled question.

When the quadric reduces to a sphere the group of automorphic collineations has the same effect upon the points of the surface as the group of geometric inversions upon the sphere. By inversion of the sphere into a plane the geometry reduces to

(B) *the inversion geometry of the plane,*

so called since its group is generated by the inversions (transformations by reciprocal radii vectores) of the plane; this group may also be defined as the totality of point transformations which leave the family of circles invariant (MÖBIUS' *Kreisverwandschaft*). The geometries (A) and (B) are equivalent in the general analytic sense which implies only the isomorphism of their groups; furthermore for the invariant methods developed in this paper they are identical.* The nomenclature employed has reference to the one or to the other geometry according as the problem is thus made clearer, but the distinction between them vanishes in the algebraic results.

Abstractly, the fundamental group of either (A) or (B) is a mixed six-parameter group G_6', consisting of two continuous systems of transformations G_6, H_6. These systems are distinguished in the (A) geometry by their effect upon the two sets of generators, the system G leaving each set invariant, while the system H interchanges the sets; and similarly in the (B) geometry by their effect upon the miminal lines or circular points at infinity. The fact that G' is a mixed group, renders its invariant theory in some respects more complicated than that of the continuous subgroup G; for in the G' theory it is not true that the sum (and *a fortiori* an integral function) of concomitants (of course homogeneous) is also a concomitant. It is therefore expedient to consider the G' theory in connection with the G theory.

The principal algebraic methods for treating the geometry (A) or (B) are based upon the following representations of the fundamental group. In the *quaternary method*, tetrahedral or tetracyclic coördinates are employed. The group G' takes the form

$$(1) \qquad G': \qquad x_i' = \sum_k r_{ik} x_k \qquad (i, k = 1, 2, 3, 4),$$

where the transformation coefficients r_{ik} satisfy the conditions which express that the fundamental quadric:

$$(2) \qquad Q \equiv \sum_{i,k} p_{ik} x_i x_k \qquad (p_{ik} = p_{ki}; \ \Delta \equiv |p_{ik}| \neq 0)$$

is, except for a factor, transformed into itself—the transformations of G being

* This does not imply that the geometries are necessarily exactly equivalent; but that the treatment of (B) by tetracyclic coördinates coincides with the treatment of (A) by tetrahedral coördinates, and similarly the treatment of (B) by minimal coördinates coincides with the treatment of (A) by generator coördinates.

distinguished by an additional relation (§ 1). In the *double binary method*, parameters $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$, are introduced in each set of generators or minimal lines, the group taking the form:

$$(3) \quad \begin{aligned} G:\ & \lambda_1' = a\lambda_1 + b\lambda_2,\ \lambda_2' = c\lambda_1 + d\lambda_2;\ \mu_1' = \alpha\mu_1 + \beta\mu_2,\ \mu_2' = \gamma\mu_1 + \delta\mu_2. \\ H:\ & \lambda_1' = a\mu_1 + b\mu_2,\ \lambda_2' = c\mu_1 + d\mu_2;\ \mu_1' = \alpha\lambda_1 + \beta\lambda_2,\ \mu_2' = \gamma\lambda_1 + \delta\lambda_2. \end{aligned}$$

In the latter coördinate system, the general algebraic curve $C_{m,n}$ upon the quadric is represented by an equation of the form:

$$\phi(\lambda_1 \overset{m}{:} \lambda_2,\ \mu_1 \overset{n}{:} \mu_2) = 0,$$

where m, n are the partial orders of the curve and indicate the number of points in which it cuts the two sets of generators. The correspondence between the curves and the double binary forms is unique; so that the (A) geometry is adequately represented by

(C) *the theory of double binary forms with respect to independent linear transformations of the variables, and also with respect to the interchange of the variables.*

This theory in connection with the (A) geometry is considered in chapter III.

In the quaternary representation, this unique correspondence between the curves and the forms does not exist. The general curve $C_{m,n}$ (where say $n = m + k$, $k \geqq 0$) requires, for its complete representation, $k + 1$ quaternary forms in point coördinates, each of degree n:

$$f_1, f_2, \cdots f_{k+1}.$$

These $k + 1$ forms are however not unique (unless $m = n = 1$); they may be any $k + 1$ linearly independent members of the linear system:

$$\nu_1 f_1 + \nu_2 f_2 \cdots + \nu_{k+1} f_{k+1} + MQ,$$

where $\nu_1 \cdots \nu_{k+1}$ are constants, and M is an arbitrary form of degree $n - 2$. It is necessary then to distinguish the quaternary theory of the (A) geometry, from

(D) *the theory of quaternary forms with respect to the groups G, G'.*

The foundations for this latter theory have been given by STUDY,* whose starting point is systems of linear forms, and final result the relation of (D) to

(E) *the theory of systems of quaternary forms including a quadric, with respect to the general linear group Γ_{15}.*

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* STUDY, *Ueber die Invarianten der projectiven Gruppe einer quadratishen Mannigfaltigkeit von nicht verschwindender Discriminante*, Leipziger Berichte, vol. 49, pp. 443–461, 1897. This paper will be referred to hereafter simply by the name of the author.

In the present paper the theory of the *forms*, i. e. (D), (E), is considered only in so far as the results are necessary for the theory of the *curves*, i. e. (A), (B). The treatment of (D) in § 1 is not to be understood as complete, for in it STUDY'S relation between (D) and (E) is assumed. The theory of the curves is taken up in §§ 4, 4′, the former considering the simplest (though most important) class, i. e., the complete intersection or equi-ordinal curves (for which $k = 0$); while the latter, with less development, treats the general curve. The final result of the quaternary method (chapter I) is the reduction of (A) to (E).

The relations between the two methods for treating the (A) or (B) geometry in the case of complete intersection curves—or more explicitly the relations between (C) on the one hand, and (D), (E) on the other—form the subject of chapter IV. The relations are of interest not merely for the (A) geometry, but also for the abstract theory of forms: they lead to principles of transference (Uebertragungsprincip) connecting the theories of quaternary and double binary forms, similar to LINDEMANN'S relations between ternary and simple binary forms. * The remaining general theory is contained in chapter VI; the methods there considered may be regarded as variations of the (C) method. In the applications, the (B) terminology is used almost exclusively, the special curves treated being the circles (chapter II) and the cyclics or bicircular quartics (chapter V).

CHAPTER I.

THE QUATERNARY METHOD.

§1. *Quaternary forms with respect to the groups G and G'.*—Each transformation g':

$$x_i' = \sum_k r_{ik} x_k ,$$

of the group G', reproduces the fundamental quadric:

$$Q = p_x^2 \qquad (\Delta = |p_{ik}| \neq 0),$$

except for a factor $D_{g'}$; so that

$$(1) \qquad \frac{\sum_{ik} p_{ik} r_{i\alpha} r_{k\beta}}{p_{\alpha\beta}} = D_{g'} \qquad (\alpha, \beta = 1, 2, 3, 4).$$

The transformation *discriminant* $D_{g'}$ is closely related to the transformation *determinant* $\delta = |r_{ik}|$, the former arising in the decomposition of the latter as follows:

* LINDEMANN, *Sur une représentation géométrique des covariantes des formes binaires*, Bulletin Société Mathématique de France, vol. 5, pp. 113–126, 1876; vol. 6, pp. 195–207, 1877. Cf. Mathematische Annalen, vol. 23, pp. 111–142, 1884.

$$|p_{ik}| \cdot |r_{ik}| = |l_{ik}|, \quad \text{where} \quad l_{ik} = \sum_{\alpha} p_{\alpha i} r_{\alpha k};$$

and

$$|l_{ik}| \cdot |r_{ik}| = |m_{ik}|,$$

where

$$m_{ik} = \sum_{\beta} l_{\beta i} r_{\beta k} = \sum_{\alpha\beta} p_{\alpha\beta} r_{\alpha i} r_{\beta k} = D p_{ik},$$

from (1). Therefore

$$\Delta\delta^2 = \Delta D^4,$$

or

$$\delta = \pm D^2,$$

the positive sign characterising the proper, and the negative the improper transformations; so that we have

(2) $$\delta_g = D_g^2, \quad \delta_h = -D_h^2.$$

In the theory of the concomitants of quaternary forms with respect to the groups G and G',—i. e., of those rational integral functions of the coefficients and variables which are unaltered (except for a factor independent of the coefficients and variables) by transformations g and g', respectively,—the preceding formulæ are fundamental. Considering for simplicity the invariants of a single form

$$f = a_x^n,$$

we have for a G invariant I,

(3) $$I(a_g) = \phi_g(r) I(a),$$

and for a G' invariant

(4) $$I(a_g) = \phi_g(r) I(a), \quad I(a_h) = \phi_h(r) I(a),$$

where a_g, a_h represent the coefficients of the transformed form, and ϕ is independent of the a's. The inverse of g and h being also members of G and H,

$$I(a) = \phi_g\left(\frac{R}{\delta}\right) I(a_g), \quad I(a) = \phi_h\left(\frac{R}{\delta}\right) I(a),$$

where R_{ik} is the minor of r_{ik} in $\delta = |r_{ik}|$; therefore in both cases

$$\phi(r)\phi\left(\frac{R}{\delta}\right) = 1 \quad \text{or} \quad \phi(r)\phi(R) = \delta^{\lambda} = \pm D^{2\lambda},$$

from (2). The discriminant D being irreducible, $\phi(r)$ is of the form cD^{μ}, say

$$\phi_g = c_1 D_g^{\mu}, \quad \phi_h = c_2 D_h^{\mu}.$$

The laws of combination of the transformations g and h give

$$c_1^2 = c_1, \; c_1 c_2 = c_2, \; c_2^2 = c_1; \quad \text{or} \quad c_1 = 1, \; c_2 = \pm 1.$$

For a G invariant the transformation factor is a power of the discriminant D^{μ}. For a G' invariant the factor produced by the transformations g is D^{μ}, while that produced by h may be either D^{μ} or $-D^{\mu}$, thus creating a division of the G' invariants into even and odd.

The theory of G and G' invariants is closely related to the ordinary theory of forms, the connection being expressed in the following theorems.

1°.—*Every projective invariant T, of*

$$Q = p_x^2, \quad f = a_x^n,$$

is a G' invariant of f.

Representing the general transformation of the projective group Γ_{15} by γ, we have by assumption,

$$T(a_\gamma, p_\gamma) = \delta_\gamma^\lambda T(a, p),$$

where λ is the weight of T. Therefore

$$T(a_{g'}, p_{g'}) = T(a_{g'}, D_{g'}p) = \delta_{g'}^\lambda T(a, p),$$

$$T(a_{g'}, p) = \frac{\delta_{g'}^\lambda}{D_{g'}^\mu} T(a, p) = \pm D_{g'}^{2\lambda-\mu} T(ap),$$

μ being the degree of T in the p's. Considered as a G' invariant, T is even or odd according as its weight λ is even or odd.

2°.—*Every sum $\sum \phi_i(p) T_i$, where the ϕ's are arbitrary functions and the T's are all of the same degree in the a's, is a G invariant; if, in addition, the weights of the T's are either all even or all odd, the sum is a G' invariant.*

This follows from 1° and from the fact that the exponent ν, of the transformation factor $\pm D^\nu$ due to each term of the sum, depends only upon the degree in the a's, while the sign depends only upon whether the weight is even or odd.

The converse of 2° has in substance been proved by STUDY * in the paper already cited. This converse is

3°.—*Every G invariant can be expressed in the form $\sum \phi_i(p) T_i$, the T's being all of the same degree in the a's; in addition, for a G invariant the weights of the T's will be either all even or all odd.*

Relations between the complete systems of concomitants for the (D) and (E) theories of the introduction, follow directly from these theorems. *A complete T system,* i. e., a system $T_1, T_2, \cdots$ such that every T can be written $T = R(T_1, T_2, \cdots)$, where R is a rational integral function with numerical coefficients,—*is also a complete G system,* i. e., every G can be written $G = R_p(T_1, T_2, \cdots)$, where R_p is a rational integral function with coefficients which may involve the p's. More generally, a system $T_1, T_2, \cdots$ such that every T can be written $T = R_p(T_1, T_2, \cdots)$, is a complete G system; conversely, from

* Leipziger Berichte, vol. 49, p. 458.

every complete G system a system of T's can be derived with the property expressed by $T = R_p(T_1, T_2, \cdots)$.

§ 2. *Systems of quaternary forms containing a quadric, with respect to the general projective group* Γ_{15}.—The simplicity of the theory of systems of forms containing a quadric, arises from the possibility of a reduction, peculiar to them, of contragrediency to cogrediency.*

In the first place *the original system S, containing a quadric and forms in any number of point and plane coördinates, may be replaced by another system S', containing only point coördinates.* For simplicity consider only one set of point coördinates x, and one of plane coördinates u. Let the system S be

$$S: \qquad Q = p_x^2, \; f = a_x^n, \; \phi = u_\alpha^m, \; \chi = b_x^r u_\beta^s .$$

By the substitution
$$u_i = p_i p_y,$$
this becomes

$$S': \qquad \begin{aligned} Q, \; f, \; \bar{\phi} &= p_\alpha p'_\alpha \cdots p_\alpha^{(m-1)} p_y p'_y \cdots p_y^{(m-1)} = A_y^m, \\ \bar{\chi} &= p_\beta \cdots p_\beta^{(s-1)} p_y \cdots p_y^{(s-1)} b_x^r = b_x^r B_y^s ; \end{aligned}$$

which in turn, by the substitution

$$y_i = P_i u_P,$$

becomes

$$S_1: \qquad \begin{aligned} Q, \; f, \; \bar{\bar{\phi}} &= A_P \cdots A_{P(m-1)} u_P \cdots u_{P(m-1)} = \Delta^m \phi, \\ \bar{\bar{\chi}} &= b_x^r B_P \cdots B_{P(s-1)} u_P \cdots u_{P(s-1)} = \Delta^s \chi . \end{aligned}$$

The system S' is composed of concomitants of S, and the system S_1, of concomitants of S'; therefore every concomitant of S' is a concomitant of S, and every concomitant of S_1 is a concomitant of S'. Again, since S_1, except for powers of Δ, coincides with S, every concomitant of S, multiplied by a suitable power of Δ, is a concomitant of S_1 and therefore also of S'. If then we regard the discriminant Δ, of the quadric Q, as a number, the systems S and S' are equivalent.

In the second place, *if again we regard Δ as a number, it is sufficient, in the study of complete form systems, to consider only in- and covariants.* For, disregarding powers of Δ, by the substitution

$$u_i = p_i p_y,$$

the totality of contravariants passes over into the totality of covariants; and mixed concomitants (i. e., those involving both point and plane coördinates) become covariants in two or more sets of congredient variables.

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* STUDY, pp. 458, 459 gives this reduction for the (D) theory; in the above the starting point is the (E) theory. Cf. the introduction.

§ 3. *The apolar surface.*—Consider the curve of intersection of the quadric $Q=0$, and the n-ic $f=0$. Of the totality of n-ic surfaces which pass through the curve, and which are therefore represented by

$$\lambda f + MQ = 0 \tag{1}$$

(λ being a constant and M an arbitrary form of degree $n-2$), there is one which is of special importance in the theory of the curve, i. e., that one which is apolar to Q considered as a class quadric,

$$\overline{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2.$$

A surface is apolar to $\overline{Q}$ when all its polar quadrics are harmonically circumscribed with respect to $\overline{Q}$, i. e., when every polar quadric is circumscribed about an infinity of tetrahedra which are self-polar with respect to $\overline{Q}$.* The covariant

$$S = a_P^2 a_x^{n-2} = \sum P_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k},$$

equated to zero, gives the locus of points whose polar quadrics are harmonic to $\overline{Q}$: its identical vanishing is therefore the necessary and sufficient condition for the apolarity of f and $\overline{Q}$.

If $F = \lambda f + MQ$ is apolar to $\overline{Q}$,

$$\Omega F = 0, \quad \text{where } \Omega = \sum P_{ik} \frac{\partial^2}{\partial x_i \partial x_k}. \tag{3}$$

It is necessary now to calculate $\Omega(MQ)$. We have

$$\frac{\partial^2}{\partial x_i \partial x_k}(MQ) = Q\frac{\partial^2 M}{\partial x_i \partial x_k} + M\frac{\partial^2 Q}{\partial x_i \partial x_k} + \frac{\partial Q}{\partial x_i}\frac{\partial M}{\partial x_k} + \frac{\partial Q}{\partial x_k}\frac{\partial M}{\partial x_i},$$

or

$$\Omega(MQ) = Q\Omega M + M\Omega Q + 2\sum P_{ik}\frac{\partial Q}{\partial x_i}\frac{\partial M}{\partial x_k};$$

but

$$\Omega Q = 2\sum P_{ik} p_{ik} = 8\Delta,$$

and

$$\sum P_{ik}\frac{\partial M}{\partial x_k}\frac{\partial Q}{\partial x_i} = 2\sum_{k,l} x_l \frac{\partial M}{\partial x_k}\sum_i P_{ik} p_{il}$$

$$= 2\Delta \sum x_i \frac{\partial M}{\partial x_i}$$

$$= 2(n-2)\Delta M.$$

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* Reye, *Ueber Algebraischen Flächen die zu einander Apolar sind*, Crelle, vol. 79, pp. 159–175, 1874.

Introducing these values, we have

$$(4)\qquad \Omega(MQ) = Q\Omega M + 4n\Delta M,$$

and by repeated application,

$$\Omega^2(MQ) = Q\Omega^2 M + 8(n-1)\Delta\Omega M,$$

$$\cdots\cdots\cdots$$

$$\Omega^h(MQ) = Q\Omega^h M + 4h(n-h+1)\Delta\Omega^{h-1}M.$$

From (2) and (3),

$$F \equiv \lambda f + MQ,$$

$$\Omega f + Q\Omega\, M + 4n\Delta M \equiv 0,$$

$$(5)\qquad \Omega^2 f + Q\Omega^2 M + 8(n-1)\Delta\Omega M \equiv 0,$$

$$\cdots\cdots$$

$$\Omega^h f + Q\Omega^h M + 4h(n-h+1)\Delta\Omega^{h-1}M \equiv 0.$$

Multiplying these equations (5) by

$$1,\ -\frac{Q}{4\cdot n},\ \frac{Q^2}{4\cdot 8\cdot n(n-1)},\ \cdots,$$

respectively, adding and putting $\lambda = \Delta^\nu$ (where ν is the greatest integer contained in $n/2$), we have the required determination,

$$(6)\qquad \begin{aligned} F = \Delta^\nu f - \frac{\Delta^{\nu-1}}{1\cdot n}\frac{Q}{4}\Omega f + \frac{\Delta^{\nu-2}}{1\cdot 2\cdot n(n-1)}\frac{Q^2}{4^2}\Omega^2 f - \cdots \\ + (-1)^\nu \frac{\Delta^\circ}{\nu!\cdot n(n-1)\cdots(n-\nu+1)}\frac{Q^\nu}{4^\nu}\Omega^\nu f. \end{aligned}$$

From its definition F is a covariant of f and Q; to put this in evidence we may express it symbolically as follows: *

$$(7)\qquad \begin{aligned} 4^\nu F = \Delta^\nu a_x^n - \frac{n(n-1)}{1\cdot n}\Delta^{\nu-1} Q a_P^2 a_x^{n-2} + \frac{n\cdots(n-3)}{1\cdot 2\cdot n(n-1)}\Delta^{\nu-2} Q^2 a_P^2 a_{P'}^2 a_x^{n-4} - \cdots \\ + (-1)^\nu \frac{n\cdots(n-2\nu+1)}{1\cdots\nu\cdot n\cdots(n-\nu+1)}\Delta^\circ Q^\nu a_P^2 a_{P'}^2 \cdots a_{P(\nu-1)}^2 a_x^{n-2\nu}. \end{aligned}$$

This determination proves that there is one and only one surface $F = 0$, which passes through the curve and is apolar to the quadric. †

* The corresponding ternary (and n dimensional) problem can be treated in the same way; it has been considered by a different method by LINDEMANN, Bulletin Société Mathématique de France, vol. 6, pp. 195–207, 1877.

† This and a more general theorem is given by STUDY, *Ueber quadratischen Formen und Linien complexe*, Leipziger Berichte, vol. 49, p. 174, 1890.

§ 4. *Equi-ordinal* * *or complete intersection curves on a quadric with reference to* G *and* G'.—The theory of the curve

$$f = 0\,, \quad Q = 0\,,$$

on the fundamental quadric, with reference to G and G' (or, what is equivalent, the inversion theory of the curve whose equation in tetracyclic coördinates connected by the quadratic equation $Q = 0$, is $f = 0$), is obviously not identical with the theory of the form f, since the latter is not completely determined by the curve. It is necessary then to *define* what is meant by a concomitant of the curve :

A G (*or* G') *concomitant of the* CURVE $f = 0$, *is a "special"* † G (*or* G') *concomitant of the* FORM $f = 0$, i. e., *a concomitant* I_f *with the property expressed by the equation :*

$$I_{\lambda f + MQ} = \psi I_f\,,$$

where ψ *is independent of the coefficients and variables involved in* I_f.

1°. The apolar form :

$$F = \Delta^{\nu} f - \frac{\Delta^{\nu-1}}{1 \cdot u}\, \frac{Q}{4}\, \Omega f + \cdots = \Delta^{\nu} f + \overline{M} Q$$

is a "special" covariant of f.

For from (6) § 3, we have $F(\lambda f + MQ) = \lambda F$.

2°. Every concomitant of F is a "special" concomitant of f. Since I is a concomitant of F it is also a concomitant of f:

$$I_F = \overline{I}_f\,.$$

Again, $\overline{I}$ is "special." For, putting $f' = \lambda f + MQ$,

$$\overline{I}_{f'} = I_{F'} = I_{\lambda F} = \lambda^{\kappa} I_F = \lambda^{\kappa} \overline{I}_f\,.$$

3°. Every "special" concomitant of f is a concomitant of F.

By assumption,

$$I_{\lambda f + MQ} = \psi I_f\,;$$

therefore

$$I_F = \psi' I_f\,,$$

$$I_f = \frac{I_F}{\psi'} = I'_F\,.$$

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* Cf. § 10 ; the general curves are considered in § 4′.

[† September 28, 1900. Cf. H. S. WHITE's definition of *semi-combinant* in his paper, *Semi-combinants as concomitants of affiliants*, American Journal of Mathematics, vol. 17, pp. 234–265, 1895. The form F of the text is a special *affiliant*. The theorems 1°, 2°, 3°, relating to G concomitants, together with their proofs, are almost identical with corresponding theorems for Γ semi-combinants given in the same paper ; in fact the former may be regarded as merely special cases of the latter.]

From these results we obtain the theorem:

The G (or G') concomitants of the curve $f=0$ coincide with the G (or G') concomitants of the apolar form F.*†

This theorem, in connection with the results of § 1, reduces the problem of finding the G or G' concomitants of the curve, to the ordinary theory of forms.

§ 4′. *The general curve.*—In the preceding section only those curves were considered which can be defined by one quaternary form in point coördinates; or, in the nomenclature of § 10, only equi-ordinal curves. A purely quaternary method for treating *all* algebraic curves is suggested in the following.

The general algebraic curve C_{mn}, on the fundamental quadric Q, whose partial orders (§ 10) are m and n (where $n = m + k$, $k \geqq 0$), may be defined by $k+1$ quaternary forms of order n,

$$f_1, f_2, \cdots f_{k+1},$$

which represent $k+1$ linearly independent n-ic surfaces through the curve. The totality of n-ic surfaces passing through the curve forms the linear system:

$$S = \nu_1 f_1 + \nu_2 f_2 + \cdots + \nu_{k+1} f_{k+1} + MQ,$$

where $\nu_1 \cdots \nu_{k+1}$ are arbitrary constants and M is an arbitrary form of order $n-2$. The definition given in § 4 may be thus generalized:‡

* By the *curve* $f=0$, is meant either the *plane* curve whose equation in tetracyclic coördinates connected by the identity $Q=0$, is $f=0$; or the *space* curve upon the quadric $Q=0$, cut out by the surface $f=0$.

† The apolar form F corresponding to the general curve $f=0$, is not the general form of degree n: the relation of F to Q causes a certain covariant of degree $10\,(n-4)$ to vanish identically. This covariant may be expressed as a determinant of the tenth order:

$$C = \begin{vmatrix} (1111) & (1112) & (1113) & (1114) & (1122) & (1123) & (1124) & (1133) & (1134) & (1144) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (4411) & (4412) & (4413) & (4414) & (4422) & (4423) & (4424) & (4433) & (4434) & (4444) \end{vmatrix},$$

where

$$(iklm) = \frac{1}{n(n-1)(n-2)(n-3)} \frac{\partial^4 f}{\partial x_i \partial x_k \partial x_l \partial x_m}.$$

The symbolic expression for this covariant given by LINDEMANN, Mathematische Annalen, vol. 23, pp. 136, 137, is incorrect; the correct expression requires the solution of the hitherto unsolved problem of the symbolic quadric through ten points.

‡ A second definition applicable to all curves, but which is not a generalization of that given in § 4, is obtained by starting from the tangential equation

$$\phi \equiv u_t^{2mn} = 0,$$

of the curve C_{mn}; or, what is equivalent, from the reciprocal of ϕ with respect to Q,

$$\bar{\phi} \equiv p_t p_t' \cdots p_t^{(2mn-1)} p_x p_x' \cdots p_x^{(2mn-1)}.$$

(The surface $\bar{\phi}=0$ cuts the fundamental quadric Q in the curve C_{mn} counted twice, and in those generators of Q which are tangent to C_{mn}). This representation suggests the definition: *the G (or G') concomitants of the curve C_{mn} are the G (or G') concomitants of the quaternary form ϕ; or, (what is equivalent except for powers of Δ), of the form $\bar{\phi}$.* Concomitants as here defined are also concomitants as defined in the text, but the converse is not true.

A G (or G') *concomitant of the curve C_{mn} is a G* (or G') *concomitant I of the forms $f_1, f_2, \cdots f_{k+1}$, which possesses the property defined by the equation:*

$$I(\sum \nu' f + M' Q, \sum \nu'' f + M'' Q, \cdots) = \psi I(f_1, f_2, \cdots),$$

where ψ is independent of the coefficients and variables involved in $I(f_1, f_2 \cdots)$.

When the curve is equi-ordinal, this definition reduces to that given in § 4. The fundamental theorem for the concomitants of the general curve is:

The G (or G') *concomitants of the curve $C_{m,n}$ are the G* (or G') *combinants* (*that is, concomitants with the combinant property*) *of the apolar forms $F_1, \cdots F_{k+1}$, corresponding to $f_1, \cdots f_{k+1}$.*

CHAPTER II.

Concomitants of Circles.

§ 5. *Quaternary system.*—Let the equations of any number of circles, in tetracyclic coördinates connected by the identity

$$Q \equiv \sum p_{ik} x_i x_k \equiv 0,$$

be

$$f_1 = A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4 = 0,$$
$$f_2 = B_1 x_1 + B_2 x_2 + B_3 x_3 + B_4 x_4 = 0, \tag{1}$$
$$f_3 = C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 = 0, \quad \text{etc.}$$

The complete system of inversion concomitants may be derived, by the method of the preceding chapter,* from the projective study of the form system

$$S: \qquad Q = p_x^2, \quad f_1 = A_x, \quad f_2 = B_x, \cdots,$$

the discriminant (of the fundamental quadric Q)

$$\Delta = \tfrac{1}{24}(pp'p''p''')^2$$

being regarded as a mere number. With this last proviso, the system S is equivalent to the system

$$S': \qquad \overline{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2, \quad f_1 = A_x, \quad f_2 = B_x, \cdots,$$

the proof being similar to that given in § 2 for the equivalence of the systems there denoted by the same letters S and S'.

Invariants of the system S' may be considered as concomitants in several sets of cogredient variables A, B, $\cdots$ of the single form $\overline{Q}$: we may there-

* In the case of circles, the concomitants of the forms (§ 1), and the concomitants of the curves (§ 4), coincide, so that no reduction by apolarity is necessary (from § 3, $F = \Delta \cdot f$).

fore apply a method of CLEBSCH,* by which the concomitants in several sets of cogredient variables are deduced from identical concomitants and polars of concomitants in one set of variables. The invariants are thus found to be of only two types:

$$A_P B_P \text{ (including the type } A_P^2),$$

and

$$(ABCD).$$

The contravariants, being invariants of S' and an additional circle, are

$$u_P^2, \quad A_P u_P, \quad (ABCD);$$

and the covariants, by the second principle of § 2, are

$$A_x, \quad (ABCp)\, p_x.$$

Collecting the preceding results one concludes:

Every G or G' *concomitant of the* n *linear forms* $f_1, f_2, \cdots, f_n$, *that is, every inversion concomitant of the* n *corresponding circles, is a rational integral function (the coefficients of which may involve the quantities* p_{ik}*) of the following concomitants:* †

$$\begin{array}{lll}
\dfrac{n(n+1)}{2} & \textit{invariants of the type} & I_{12} = A_P B_P, \\[2ex]
\dbinom{n}{4} & \quad\text{"} \quad \text{"} \quad \text{"} \quad \text{"} & K_{1234} = (ABCD), \\[2ex]
n & \textit{covariants} \text{ " " "} & f_1 = A_x, \\[2ex]
\dbinom{n}{3} & \quad\text{"} \quad \text{"} \quad \text{"} \quad \text{"} & g_{123} = (ABCp)p_x, \\[2ex]
n & \textit{contravariants of the type} & F_1 = A_P u_P, \\[2ex]
\dbinom{n}{3} & \quad\text{"} \quad \text{"} \quad \text{"} \quad \text{"} & G_{123} = (ABCu), \\[2ex]
\multicolumn{2}{l}{\textit{and the identical contravariant}} & Q = u_P^2.
\end{array}$$

The non-symbolic expressions for these forms, in the case of the orthogonal system of tetracyclic coördinates $(Q \equiv \sum x^2)$, are

$$f_1 = \sum A_i x_i, \quad F_1 = \sum A_i u_i, \quad I_{12} = \sum A_i B_i,$$
$$K_{1234} = (ABCD), \quad g_{123} = (ABCx), \quad G_{123} = (ABCu).$$

* CLEBSCH, *Ueber ein fundamentale Aufgabe der Invariantentheorie,* Göttingen Abhandlungen, vol. 17, p. 39, 1872.

† These (or rather equivalent) forms are determined directly by STUDY (p. 443 ff.), i. e., without passing to the ordinary theory of invariants; the same is true of the quaternary syzygies of § 7.

§ 6. *Binary system.*—In minimal coördinates * λ, μ, the equation of a circle is

$$a_{11}\lambda\mu + a_{12}\lambda + a_{21}\mu + a_{22} = 0,$$

or in homogeneous form,

$$f_1 \equiv a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = 0.$$

A circle is therefore represented by a binary bilinear form with non-cogredient variables. Such forms have been studied by PEANO.† If the n circles be represented by the symbols of their bilinear forms,

$$f_1 = a_\lambda a_\mu, \quad f_2 = b_\lambda \beta_\mu, \quad f_3 = c_\lambda \gamma_\mu, \cdots,$$

their complete binary system is composed of the following forms:

$$(1) \qquad \begin{array}{ccll}
n & \text{of type} & f_1 & = a_\lambda a_\mu, \\
n & \text{"} & F_1 & = (ar)(a\rho), \\
\dfrac{n(n+1)}{2} & \text{"} & I_{12} & = (ab)(a\beta), \\
\dbinom{n}{4} & \text{"} & K_{1234} & = (ac)(bd)(a\beta)(\gamma\delta) - (ab)(cd)(a\gamma)(\beta\delta), \\
\dbinom{n}{3} & \text{"} & g_{123} & = (ac)(a\beta)b_\lambda\gamma_\mu - (ab)(a\gamma)c_\lambda\beta_\mu, \\
\dbinom{n}{3} & \text{"} & G_{123} & = (ac)(a\beta)(br)(\gamma\rho) - (ab)(a\gamma)(cr)(\beta\rho), \\
\dbinom{n}{2} & \text{"} & L_{12} & = (a\beta)a_\lambda b_\mu, \\
\dbinom{n}{2} & \text{"} & M_{12} & = (ab)a_\mu\beta_\mu, \\
1 & \text{"} & \bar{Q} & = (rr')(\rho\rho'),
\end{array}$$

where the variable circle occurring in the contravariants is $r_{11}\lambda_1\mu_1 + r_{12}\lambda_1\mu_2 + r_{21}\lambda_2\mu_1 + r_{22}\lambda_2\mu_2 = r_\lambda\rho_\mu = r'_\lambda\rho'_\mu$. This system may be deduced by PEANO's direct method; it may also be obtained (except for L and M) by a transformation of the quaternary results of § 5. (Cf. chapter IV.)

* These are discussed in § 10. The simplest element in these coördinates is not the circle, but the minimal line, whose equation is of the form $a_1\lambda_1 + a_2\lambda_2 = 0$, or $a_1\mu_1 + a_2\mu_2 = 0$. The complete system of any number of such lines a_λ, b_λ, $\cdots$, a_μ, β_μ $\cdots$, consists of the forms themselves and of the invariants (ab), $(a\beta)$, etc.

† PEANO, *Formazioni invariantive della correspondenza,* Giornale di Matematiche, vol. 20, pp. 81-88, 1882.

The non-symbolic values of the concomitants (1) are

$$f_1 = a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 ,$$

$$F_1 = a_{11} r_{22} - a_{12} r_{21} - a_{21} r_{12} + a_{22} r_{22} ,$$

$$I_{11} = 2a_{11} a_{22} - 2a_{12} a_{21} , \quad I_{12} = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} ,$$

$$K_{1234} = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ d_{11} & d_{12} & d_{21} & d_{22} \end{vmatrix} , \quad G_{123} = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ r_{11} & r_{12} & r_{21} & r_{22} \end{vmatrix} ,$$

$$g_{123} = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_2\mu_2 \end{vmatrix} , \quad \overline{Q} = 2r_{11}r_{22} - 2r_{12}r_{21} ,$$

$$L_{12} = (a_{11}b_{12} - a_{12}b_{11})\lambda_1^2 + (a_{11}b_{22} - a_{12}b_{21} + a_{21}b_{12} - a_{22}b_{11})\lambda_1\lambda_2 + (a_{21}b_{22} - a_{22}b_{21})\lambda_2^2 ,$$

$$M_{12} = (a_{11}b_{21} - a_{21}b_{11})\mu_1^2 + (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})\mu_1\mu_2 + (a_{12}b_{22} - a_{22}b_{12})\mu_2^2 .$$

§ 7. *Syzygies in the preceding systems.*—A series of simple considerations shows that the systems given in §§ 1, 2, are not only complete, but also irreducible; however the members of the systems are not independent. For the quaternary system the relations among the invariants are*

$$(1) \qquad \Delta^3 K_{\alpha\beta\gamma\delta} K_{\alpha'\beta'\gamma'\delta'} = \begin{vmatrix} I_{\alpha\alpha'} & I_{\alpha\beta'} & I_{\alpha\gamma'} & I_{\alpha\delta'} \\ I_{\beta\alpha'} & I_{\beta\beta'} & I_{\beta\gamma'} & I_{\beta\delta'} \\ I_{\gamma\alpha'} & I_{\gamma\beta'} & I_{\gamma\gamma'} & I_{\gamma\delta'} \\ I_{\delta\alpha'} & I_{\delta\beta'} & I_{\delta\gamma'} & I_{\delta\delta'} \end{vmatrix} ,$$

$$(2) \qquad I_{i\alpha}K_{\beta\gamma\delta\epsilon} + I_{i\beta}K_{\gamma\delta\epsilon\alpha} + I_{i\gamma}K_{\delta\epsilon\alpha\beta} + I_{i\delta}K_{\epsilon\alpha\beta\gamma} + I_{i\epsilon}K_{\alpha\beta\gamma\delta} = 0 .$$

The relation (1) is proved by decomposing the right hand member, after substituting the symbolic values given in §5, as follows:

$$\begin{vmatrix} A_P A'_P & A_P B'_P & A_P C'_P & A_P D'_P \\ B_{P'} A'_{P'} & B_{P'} B'_{P'} & B_{P'} C'_{P'} & B_{P'} D'_{P'} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \end{vmatrix}$$

* STUDY, l. c., p. 444, gives relations equivalent to (1) and (2), and also proves that they form the complete system of syzygies.

$$= A_P B_{P'} C_{P''} D_{P'''} \begin{vmatrix} A'_P & B'_P & C'_P & D'_P \\ A'_{P'} & B'_{P'} & C'_{P'} & D'_{P'} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \tfrac{1}{24} \begin{vmatrix} A_P & B_P & C_P & D_P \\ A_{P'} & B_{P'} & C_{P'} & D_{P'} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \cdot \begin{vmatrix} A'_P & B'_P & C'_P & D'_P \\ A'_{P'} & B'_{P'} & C'_{P'} & D'_{P'} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \tfrac{1}{24}(ABCD)(A'B'C'D')(PP'P''P''')^2$$
$$= \Delta^3(ABCD)(A'B'C'D').$$

The relation (2) is proved by rewriting the first member as follows:

$$F_P \begin{vmatrix} A_P & B_P & C_P & D_P & E_P \\ A_1 & B_1 & C_1 & D_1 & E_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_4 & B_4 & C_4 & D_4 & E_4 \end{vmatrix}.$$

The relations among the remaining forms of the quaternary system are of the same two types. Only a few, which are to be employed later, will here be written out.

In (1) let f_δ and $f_{\delta'}$ be the same degenerate circle, i. e., point circle; then

(3)
$$\Delta^3 g_{\alpha\beta\gamma} g_{\alpha'\beta'\gamma'} = \begin{vmatrix} I_{\alpha\alpha'} & I_{\alpha\beta'} & I_{\alpha\gamma'} & f_\alpha \\ I_{\beta\alpha'} & I_{\beta\beta'} & I_{\beta\gamma'} & f_\beta \\ I_{\gamma\alpha'} & I_{\gamma\beta'} & I_{\gamma\gamma'} & f_\gamma \\ f_{\alpha'} & f_{\beta'} & f_{\gamma'} & 0 \end{vmatrix}.$$

Again, let f_i in (2) become a point; then

(4)
$$f_\alpha K_{\beta\gamma\delta\epsilon} + f_\beta K_{\gamma\delta\epsilon\alpha} + f_\gamma K_{\delta\epsilon\alpha\beta} + f_\delta K_{\epsilon\alpha\beta\gamma} + f_\epsilon K_{\alpha\beta\gamma\delta} = 0.$$

Another important syzygy among the invariants is

(5)
$$\begin{vmatrix} I_{\alpha\alpha'} & I_{\alpha\beta'} & I_{\alpha\gamma'} & I_{\alpha\delta'} & I_{\alpha\epsilon'} \\ I_{\beta\alpha'} & I_{\beta\beta'} & I_{\beta\gamma'} & I_{\beta\delta'} & I_{\beta\epsilon'} \\ I_{\gamma\alpha'} & I_{\gamma\beta'} & I_{\gamma\gamma'} & I_{\gamma\delta'} & I_{\gamma\epsilon'} \\ I_{\delta\alpha'} & I_{\delta\beta'} & I_{\delta\gamma'} & I_{\delta\delta'} & I_{\delta\epsilon'} \\ I_{\epsilon\alpha'} & I_{\epsilon\beta'} & I_{\epsilon\gamma'} & I_{\epsilon\delta'} & I_{\epsilon\epsilon'} \end{vmatrix} = 0,$$

which is however not distinct from (1) and (2), for the above determinant may be written:

$$\Delta^3 K_{\alpha'\beta'\gamma'\delta'}(I_{\epsilon'\alpha}K_{\beta\gamma\delta\epsilon} + I_{\epsilon'\beta}K_{\gamma\delta\epsilon\alpha} + \cdots).$$

The above relations apply also to the binary system, if in them we put $\Delta = 1$; in addition,*

$$-2L_{\alpha\beta}M_{\gamma\delta} = I_{\alpha\gamma}f_\beta f_\delta - I_{\alpha\delta}f_\beta f_\gamma - I_{\beta\gamma}f_\alpha f_\delta + I_{\beta\delta}f_\alpha f_\gamma + f_\gamma g_{\alpha\delta\beta} - f_\alpha g_{\beta\gamma\delta}.$$

§ 8. *Geometric interpretations.*†—Two systems of concomitants of circles have been given in §§ 6, 7. The first applies also to a quadric and a set of planes, the second also to a set of binary homographies in which the carriers are not regarded as superposed: for each of these points of view there is a corresponding geometric interpretation which will not, however, be given here.

If then only circles in the plane be considered, the interpretations of the vanishing of the concomitants are as follows:

$f_1 = 0$, represents a circle; and

$F_1 = 0$, the linear complex of circles orthogonal to f_1.

$I_{11} = 0$, is the condition for the degeneration of the circle f; and

$I_{12} = 0$, is the condition for the orthogonality of f_1 and f_2.

$K_{1234} = 0$, is the condition that the four circles f_1, f_2, f_3, f_4 belong to a linear system, i. e., have a common orthogonal circle.

$g_{123} = 0$, represents the circle orthogonal to f_1, f_2, f_3; and

$G_{123} = 0$, represents the linear complex orthogonal to g_{123}.

$\bar{Q} = 0$, represents the quadratic complex consisting of all the degenerate circles of the plane.

The preceding forms are common to both systems; in the binary system we have, in addition, L_{12}, M_{12}, which represent the pairs of minimal lines of each series through the intersections of f_1 and f_2.

All invariant relations of circles must be representable through the above concomitants; some examples will now be considered.

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* Peano, l. c.

† On the geometry of circles see Study, *Das Apollonische Problem*, Mathematische Annalen, vol. 49, 1897.

Möbius, *Kreisverwandschaft*, Werke, vol. II, 1852.

Loria, *Geometria della sfera*, Memorie Accademia Torino, vol. 36, 1885.

Lachlan, Philosophical Transactions of the Royal Society, 1886.

The condition for the tangency of two circles f_1, f_2 is

$$\begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix} = 0 ;$$

and the condition that three circles f_1, f_2, f_3, have a common point is

$$\begin{vmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{vmatrix} = 0 .$$

These are combinants applying equally to any two or three members of the linear systems $t_1 f_1 + t_2 f_2$, $t_1 f_1 + t_2 f_2 + t_3 f_3$, respectively.

The circles cutting f_1, f_2 in four points whose anharmonic ratio is a, are given by

$$\left(\frac{a+1}{a-1}\right)^2 = \frac{(I_{12}\sigma' - F_1 F_2)^2}{(I_{11}\sigma' - F_1^2)(I_{22}\sigma' - F_2^2)} ,$$

the harmonic circles (i. e., $a = -1$) being

$$I_{12}\sigma' - F_1 F_2 = 0 .$$

As an application, let it be required to find the equation, in invariant form, of the cyclic curve* through eight given points †. The circle through the points 1, 2, 3, is $g_{123} = 0$; the system of circles through six of the points is therefore:

$$t_1 g_{123} g_{456} + t_2 g_{124} g_{356} + t_3 g_{125} g_{346} = 0 .$$

If we impose the conditions for the curve going through the remaining two points, and eliminate t_1, t_2, t_3, the required cyclic is found to be

$$\begin{vmatrix} g_{123}\, g_{456} & g_{124}\, g_{356} & g_{125}\, g_{346} \\ K_{1237} K_{4567} & K_{1247} K_{3567} & K_{1257} K_{3467} \\ K_{1238} K_{4568} & K_{1248} K_{3568} & K_{1258} K_{3468} \end{vmatrix} = 0 .$$

§ 9. *Absolute invariants.*—Two circles have a single absolute invariant:

$$C_{12} = \frac{I_{12}^2}{I_{11} I_{22}} = \cos^2 \theta_{12} ,$$

* Otherwise bicircular biquadratic.

† Points may be considered as degenerate circles, so that the concomitants apply also to sets of points. The condition that two points lie on a minimal line is $I_{12} = 0$; the condition that four points are cocircular is $K_{1234} = 0$; etc.

where θ_{12} is the angle of the circles. All the absolute invariants of any number of circles are functions of the quantities

$$C_{ik} = \frac{I_{ik}}{I_{ii} I_{kk}} = \cos^2 \theta_{ik},$$

θ_{ik} denoting the angle of the circles f_i, f_k; but not every rational absolute invariant is a rational function of the C_{ik}'s. To obtain systems of absolute invariants possessing the latter property—which systems may be called complete *rational* systems—it is necessary to introduce the absolute invariants

$$(1) \qquad D_{iklm} = \frac{I_{il} I_{km}}{I_{im} I_{kl}}, \quad E_{iklm} = \frac{K_{iklm}}{I_{ik} I_{lm}}.$$

Every rational absolute invariant of any number of circles, with respect to the mixed inversion group G', *is a rational function of the quantities* D_{iklm}; *with respect to the continuous inversion group* G, *every rational absolute invariant is a rational function of the quantities* E_{iklm}.

Consider now the case where the circles reduce to points. *Four points* $P_1 P_2 P_3 P_4$ *have two absolute invariants*

$$(2) \qquad \alpha = D_{1234}, \quad \beta = D_{1324},$$

which completely characterize their invariant properties with respect to the group G'. With respect to the group G, the simplest absolute invariants present themselves only in the minimal system of coördinates. Let the coördinates of P_i be a_i, b_i; then the fundamental absolute invariants of the four points are

$$(3) \qquad \sigma = \frac{(a_1 - a_3)(a_2 - a_4)}{(a_1 - a_4)(a_2 - a_3)}, \quad \tau = \frac{(b_1 - b_3)(b_2 - b_4)}{(b_1 - b_4)(b_2 - b_3)}.$$

The relations between (2) and (3) are

$$\alpha = \sigma\tau, \quad \beta = (1 - \sigma)(1 - \tau).$$

If the points be taken in all possible orders, six distinct pairs of values for α, β and σ, τ are found:

$$(4) \qquad \begin{array}{lll} 1234: & \alpha, \beta; & \sigma, \tau; \\ 1324: & \beta, \alpha; & 1-\sigma, 1-\tau; \\ 1243: & \frac{1}{\alpha}, \frac{\beta}{\alpha}; & \frac{1}{\sigma}, \frac{1}{\tau}; \\ 1342: & \frac{1}{\beta}, \frac{\alpha}{\beta}; & \frac{1}{1-\sigma}, \frac{1}{1-\tau}; \\ 1423: & \frac{\beta}{\alpha}, \frac{1}{\alpha}; & \frac{\sigma-1}{\sigma}, \frac{\tau-1}{\tau}; \\ 1432: & \frac{\alpha}{\beta}, \frac{1}{\beta}; & \frac{\sigma}{\sigma-1}, \frac{\tau}{\tau-1}. \end{array}$$

With respect to either group G or G', the general set of four points is equivalent to itself in four ways. *The "special" sets,* i. e., *those which are equivalent to themselves in more than four ways,** are as follows:

First, as to the group G:

1°. $\sigma, \tau = 0, 0$ or $1, 1$ or ∞, ∞; equivalent to itself in 8 ways. This is the *coincident* set.

2°. $\sigma, \tau = -1, -1$ or $2, 2$ or $\frac{1}{2}, \frac{1}{2}$; equivalent to itself in 8 ways. This is the *harmonic* set.

3°. $\sigma, \tau = \omega, 1-\omega$ or $1-\omega, \omega$ where $\omega^3 = -1$; 12 ways. This is the *equianharmonic* set.

4°. $\sigma, \tau = \frac{1}{2}, \infty$ or $2, 0$ or $-1, 1$; 8 ways.

5°. $\sigma, \tau = \omega, 1-\omega$ or $1-\omega, \omega$ where $\omega^3 = -1$; 12 ways.

Second, as to the group G':

A. $\alpha, \beta = 1, \lambda$ or $\lambda, 1$ or $1/\lambda, 1/\lambda$ where $\lambda \neq 1$; 8 ways. This includes cases 1° (for which $\lambda = 0$) and 2° (for which $\lambda = 4$).

B. $\alpha, \beta = 1, 1$; 24 ways. This coincides with case 3°.

C. $\alpha, \beta = 0, -1$ or $-1, 0$ or ∞, ∞; 8 ways. This coincides with case 4°.

D. $\alpha, \beta = \epsilon, \epsilon^2$ or ϵ^2, ϵ where $\epsilon^3 = 1$; 12 ways. This coincides with case 5°.

The geometrical definitions and properties of these special sets present considerable interest, but will not be given here.

CHAPTER III.

THE BINARY METHOD.

§ 10. *Minimal coördinates.*—In the system discussed in Chapter I (tetracyclic coördinates in the plane, or tetrahedral in space), not every algebraic curve can be represented by a single equation in connection with the identity; for such an equation represents a curve cutting the minimal lines (or generators) of each system in the same number of points.† Calling the numbers which indicate how many times the lines of the two systems cut an (algebraic) curve, the partial orders of the curve, we see that the quaternary theory as developed in § 4 is restricted to the domain of "equi-ordinal" curves.‡

* It is to be noticed that this is not identical with the statement that the points admit more than four automorphic transformations: the text classification is based on the number of *equivalences*, not on the number of *transformations*. (In a classification based upon the latter, the cocircular set of four points would count as special, since such a set admits eight tranformations.)

† Conversely, every equi-ordinal curve can be represented by a single tetracyclic equation. For the general theory see KLEIN, *Ein liniengeometrischen Satz*, Göttingen Nachrichten, 1872; or Mathematische Annalen, vol. 22, pp. 234–241, 1882.

‡ This is not a necessary restriction of the system of tetracyclic coördinates, but only of the apolar method developed in § 4. Cf. § 4′, for curves whose partial orders are unequal.

This limitation is removed more conveniently than in § 4′ by the use of what may be termed the minimal system of coördinates.* In this, a point is determined by two coördinates λ, μ, the parameters of the pair of minimal lines (or generators) through the point—the parameters being ordinary projective coördinates in the minimal pencils (or hyperpencils of generators). Instead of λ, μ, we may introduce two pairs of homogeneous coördinates $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$.

A curve whose partial orders are m, n, is represented by an equation $f(\overset{m}{\lambda}, \overset{n}{\mu}) = 0$, of degree m in λ and degree n in μ. Introducing the homogeneous coördinates, the equation of the general algebraic curve may be written

$$(1) \qquad f = f(\lambda_1, \lambda_2; \mu_1, \mu_2) = \sum_{h,k} \binom{m}{h}\binom{n}{k} a_{hk} \lambda_1^{m-h} \lambda_2^h \mu_1^{n-k} \mu_2^k = 0,$$

h and k varying from 0 to m and from 0 to n respectively. Symbolically, we may write

$$(2) \qquad f = a_\lambda^m a_\mu^n = b_\lambda^m \beta_\mu^n = \cdots,$$

where the symbols have real meaning only in the combinations :

$$a_1^{m-h} a_2^h a_1^{n-k} a_2^k = b_1^{m-h} b_2^h \beta_1^{n-k} \beta_2^k = \cdots = a_{hk}.$$

The inversion group G_6' in the present coördinates takes the form :

$$(3) \qquad \begin{aligned} G_6 &: \qquad L = \frac{a\lambda + b}{c\lambda + d}, \quad M = \frac{\alpha\mu + \beta}{\gamma\mu + \delta}; \\ H_6 &: \qquad L = \frac{a\mu + b}{c\mu + d}, \quad M = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}. \end{aligned}$$

In the present chapter, however, attention will be restricted † to the continuous group G_6, which may also be written

$$(4) \quad G_6 : \quad L_1 = a\lambda_1 + b\lambda_2, \; L_2 = c\lambda_1 + d\lambda_2, \; M_1 = \alpha\mu_1 + \beta\mu_2, \; M_2 = \gamma\mu_1 + \delta\mu_2.$$

§ 11. *Double binary forms.*—From (2) and (4) we have:

The invariant theory of curves for the group G_6, *is equivalent to the theory of double binary forms whose variables undergo independent linear transformations.*

The theory of double binary forms in which the variables undergo the *same* linear transformation, may be regarded as completely known ; for we may pass

* The system is fixed by three base points $P'P''P'''$: denoting the corresponding minimal lines by $L'L''L'''M'M''M'''$, the coördinates of a point P are the anharmonic ratios $\lambda = (LL'L''L''')$, $\mu = (MM'M''M''')$. It has a theoretic advantage over the tetracyclic system, in that it introduces only invariant elements in its definition.

† See § 21 for the group G'.

from the consideration of such forms, immediately to an equivalent system of simple binary forms. In fact the concomitants of the form $r_x^m s_y^n$ (where x and y are cogredient), are the concomitants of the set: *

$$r_x^m s_x^n, \quad (rs) r_x^{m-1} s_x^{n-1}, \quad (rs)^2 r_x^{m-2} s_x^{n-2}, \quad \cdots.$$

Where, however, the variables, as in $a_\lambda^m a_\mu^n$, are not cogredient, but undergo distinct transformations, the theory is not so simple; methods, analagous to those for simple binary forms, have been developed by PEANO and GORDAN † for determining the concomitants, and it has been proved that their totality may be represented by a complete system. In the following sections a geometric basis is given to these methods, principally by the consideration of the polar theory of the forms and curves.

§ 12. *Polar theory of the general curve* $C_{m,n}$.—The double binary form

$$f = a_\lambda^m a_\mu^n,$$

equated to zero, represents a curve cutting each λ-line in n points, and each μ-line in m points—where by a λ-line is meant a minimal line for which λ is constant, etc. Such a curve will be denoted by $C_{m,n}$; so that a λ-line is a $C_{1,0}$, a μ-line a $C_{0,1}$, a circle a $C_{1,1}$, a cyclic curve a $C_{2,2}$, etc. The general $C_{m,n}$ contains $mn + m + n$ independent constants; its total order, i. e., the number of points in which it is cut by an arbitrary circle, is $m + n$. Two curves $C_{m,n}$, $C_{m',n'}$ intersect in $mn' + nm'$ points.‡ All $C_{m,n}$'s which have $mn + m + n$ points in common, will in general also have $mn - m - n$ additional points in common.

Passing over the general properties of curves, of which the above are simple types, consider now the (inversion) polars of a curve $C_{m,n}$. *The* (h, k) *polar* $P_{h,k}$ *of a point* ξ, η *with respect to a curve* $f \equiv a_\lambda^m a_\mu^n = 0$, *is defined thus:*

$$P_{h,k} \equiv a_\xi^h a_\eta^k a_\lambda^{m-h} a_\mu^{n-k} = 0; \tag{2}$$

which may also be written

$$P_{h,k} = \Delta_1^h \Delta_2^k f,$$

where

$$\Delta_1 = \frac{1}{m}\left(\xi_1 \frac{\partial}{\partial \lambda_1} + \xi_2 \frac{\partial}{\partial \lambda_2}\right), \qquad \Delta_2 = \frac{1}{n}\left(\eta_1 \frac{\partial}{\partial \mu_1} + \eta_2 \frac{\partial}{\partial \mu_2}\right). \tag{3}$$

* CLEBSCH, Göttingen Abhandlungen, vol. 17, p. 23; *Binäre Formen*, § 14.

† PEANO, Atti Accademia Torino, 1881; Giornale di Matematiche, vol. 20, pp. 79-101, 1882. GORDAN, Mathematische Annalen, vol. 23, pp. 372-389, 1889.

‡ The general $C_{m,n}$, from the standpoint of projective geometry, is the special curve of order $m + n$, (ϕ_{m+n}), which has multiple points of orders m, n at the circular points I, J, respectively; while from the point of view of inversion geometry, the general curve of order κ, (ϕ_κ), is the special $C_{\kappa,\kappa}$, which has a multiple point of order κ at infinity. Projectively, a C_{mn} and a $C_{m'n'}$ cut in $(m+n)(m'+n')$ points; but of these the mm' at I and the nn' at J are discarded in inversion geometry.

In expanded form,

$$(4)\qquad P_{h,k}=\sum_{i,j}\binom{h}{i}\binom{h}{j}\xi_1^{h-i}\xi_2^i\eta_1^{k-j}\eta_2^j f_{h-i,\,i,\,k-j,\,j}(\lambda\,,\ \mu)\qquad \begin{pmatrix} i=0,\ 1,\cdots m\\ k=0,\ 1,\cdots n\end{pmatrix},$$

where

$$f_{abcd}=\frac{1}{m\cdots(m-a-b-1)n\cdots(n-c-d-1)}\cdot\frac{\partial^{a+b+c+d}f}{\partial\lambda_1^a\partial\lambda_2^b\partial\mu_1^c\partial\mu_2^d}.$$

The $(h\,,\ k)$ polar may also be called the polar $C_{m-h,\ n-k}$; this is put in evidence by introducing for it the double notation $P_{h,\,k}\equiv Q_{m-h,\ n-k}$.

The polar $C_{h,\,k}$ of a point $\xi\,,\ \eta$ is therefore:

$$Q_{h,\,k}=\sum_{i,j}\binom{h}{i}\binom{k}{j}f_{h-i,\,i,\,k-j,\,j}(\xi\,,\ \eta)\lambda_1^{h-i}\lambda_2^i\mu_1^{k-j}\mu_2^j\,.$$

The geometric definition of the polar curves may be obtained from the consideration of simple binary polarity in each of the minimal pencils. The curve $f=a_\lambda^m a_\mu^n$ may be regarded as establishing an $(m\,,\ n)$ correspondence between the system of λ-lines and the system of μ-lines: to each λ-line there corresponds the n μ-lines $\mu_1\mu_2\cdots\mu_n$ through the intersections of the λ-line with f, and to each μ-line the m λ-lines $\lambda_1\cdots\lambda_m$ through the intersections of the μ-line with f. Similarly with respect to the curve,

$$(5)\qquad a_\mu^{n-k}a_\lambda^m a_\eta^k=0\,,$$

to each λ corresponds the kth polar of η as to $\mu_1\mu_2\cdots\mu_n$; there are m λ's, $\lambda_1'\lambda_2'\cdots\lambda_m'$, such that with respect to the μ's corresponding to each in f, μ belongs to the kth polar of η. Passing from (5) to

$$a_\lambda^{m-h}a_\xi^h a_\mu^{n-k}a_\eta^k=0\,,$$

we obtain the required geometric definition:

The $(h\,,k)$ polar of a point $\xi\,,\eta$ with respect to a curve $C_{m,\,n}$ is a curve $Q_{m-h,\ n-k}$, generated by the following correspondence between the minimal pencils: to each μ correspond those λ's *$(m-h$ in number$)$ which form the h-th polar of ξ with respect to $\lambda_1'\lambda_2'\cdots\lambda_m'$—where the λ''s are defined by the fact that the μ's corresponding to each in f are such that, with respect to them, μ belongs to the k-th polar of η; and similarly to each λ, etc.*

Of special importance are the polars Q_{10}, Q_{01}, Q_{11}. The polar minimal lines of a point P are

$$Q_{10}=a_\xi^{m-1}a_\eta^n a_\lambda=f_{10}\lambda_1+f_{20}\lambda_2\,,$$

and

$$Q_{01}=a_\xi^m a_\eta^{n-1}a_\mu=f_{01}\mu_1+f_{02}\mu_2\qquad\left(f_{ik}=\frac{1}{mn}\,\frac{\partial^2 f}{\partial\lambda_i\partial\mu_k}\right);$$

Q_{10} is the linear polar of ξ with respect to the λ's corresponding to η in f, and Q_{01} the linear polar of η with respect to the μ's corresponding to ξ in f. To each point P there corresponds a definite conjugate point P', the intersection of the polar minimal lines Q_{10}, Q_{01}. The number of points P which have a given point P' for conjugate is

$$mn + (m-1)(n-1) = 2mn - m - n + 1\,;$$

only in the case of the circle* $(m = 1,\ n = 1)$, therefore, is the conjugate relation involutorial.

The polar circle of P is,

$$Q_{11} = a_\xi^{m-1} a_\eta^{n-1} a_\lambda a_\mu = f_{11}\lambda_1\mu_1 + f_{12}\lambda_1\mu_2 + f_{21}\lambda_2\mu_1 + f_{22}\lambda_2\mu_2\,.$$

It is generated by a $(1,\ 1)$ correspondence: to each μ corresponds the linear polar of ξ with respect to $\lambda_1' \cdots \lambda_m'$, where the λ''s are such that with respect to the μ's corresponding in f, μ is the linear polar of η.

Between the polar minimal lines, the polar circles, and the conjugate point of any point P with respect to f, there exists this relation: the pair of lines through the given point is cut in the same two points by the polar circle and by the polar minimal lines; or the conjugate of P is the inverse of P with respect to the polar circle.

The general properties of the polar curves may be developed by practically the same methods as in the projective theory. If the point B lies on the $(h,\ k)$ polar of the point A, A lies on the $(m-h,\ n-k)$ polar of B. The polar C_{hk} of P with respect to f is also the polar C_{hk} of P with respect to any polar of f. The (hk) polar of P with respect to the $(h'k')$ polar of P is the $(h+h',\ k+k')$ polar of P. If P lies on f, all its polars are tangent to f at P, etc.

§ 13. *Transvectants.*—In the theory of simple binary forms, the most important process for the formation of invariants is that of transvection (Ueberschiebung): CLEBSCH and GORDAN have shown that this single process, by repeated application, will yield the totality of concomitants. An analogous process of double transvection (doppio scorrimento), has been applied by PEANO† to the double binary forms.

Let any two forms (not necessarily distinct) be

$$f = a_\lambda^m a_\mu^n\,, \quad \phi = b_\lambda^{m'} \beta_\mu^{n'}\,;$$

their (hk) *double transvectant* is defined by

$$(1) \qquad (f\phi)_{hk} = (ab)^h (a\beta)^k a_\lambda^{m-h} b_\lambda^{m'-h} a_\mu^{n-k} \beta_\mu^{n'-k}\,.$$

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* In this case the relation of conjugacy coincides with inversion.

† PEANO, *Formazioni invariantive della correspondenza*, Giornale di Matematiche, vol. 20, p. 79, 1882.

The non-symbolic expression is

$$(2)\qquad (f\phi)_{hk} = [\Omega_1^h \Omega_2^k f(\lambda\mu)\phi(\xi\eta)]_{\xi,\,\eta=\lambda,\,\mu},$$

where Ω_1, Ω_2 are the operators defined by *

$$\Omega_1 = \frac{1}{mm'}\left(\frac{\partial^2}{\partial\lambda_1\partial\xi_2} - \frac{\partial^2}{\partial\lambda_2\partial\xi_1}\right), \quad \Omega_2 = \frac{1}{nn'}\left(\frac{\partial^2}{\partial\mu_1\partial\eta_2} - \frac{\partial^2}{\partial\mu_2\partial\eta_1}\right).$$

Transvectants are covariants which are linear in the coefficients of each form. If f and ϕ have the same orders m, n, the (mn) transvectant is an invariant

$$(f\phi)_{mn} = (ab)^m(\alpha\beta)^n,$$

which, as in the case of simple binary forms, may be called the bilinear, harmonic or apolar invariant of the forms.

THEOREM. *The locus of points whose polar $C_{h,k}$'s with respect to f and ϕ are apolar is $(f\phi)_{hk} = 0$.*

For the polar $C_{h,k}$'s are

$$Q^f_{hk} = a_\xi^{m-h}\alpha_\eta^{n-k}a_\lambda^h\alpha_\mu^k,$$

$$Q^\phi_{hk} = b_\xi^{m'-h}\beta_\eta^{n'-k}b_\lambda^h\beta_\mu^k;$$

therefore their apolar invariant is

$$(Q^f_{hk},\ Q^\phi_{hk})_{hk} = (ab)^h(\alpha\beta)^k a_\xi^{m-h}b_\xi^{m'-h}\alpha_\eta^{n-k}\beta_\eta^{n'-k}.$$

When f and ϕ coincide, the theorem becomes: the locus of points whose polar C_{hk}'s with respect to f are self-apolar is $(ff)_{hk} = 0$.

Consider now the simplest transvectants, which play the rôle here of Jacobians and Hessians in projective geometry:

$$(f\phi)_{10} = (ab)a_\lambda^{m-1}b_\lambda^{m'-1}\alpha_\mu^n\beta_\mu^{n'} = f_{10}\phi_{20} - f_{20}\phi_{10},$$

$$(f\phi)_{01} = (\alpha\beta)a_\lambda^m b_\lambda^{m'}\alpha_\mu^{n-1}\beta_\mu^{n'-1} = f_{01}\phi_{02} - f_{02}\phi_{01}.$$

The curve $(f\phi)_{10} = 0$ is the locus of points whose polar λ-lines with respect to f and ϕ coincide; for the points of $(f\phi)_{01} = 0$ the polar μ lines coincide.

If a point lies on both these transvectants, it has the same minimal lines with respect to each of the curves, therefore:

There are $2(m+m')(n+n') - 2(m+n+m'+n') + 4$ points whose conjugate points with respect to f and ϕ coincide; these are the intersections of $(f\phi)_{10}$ and $(f\phi)_{01}$, and include the $mn' + m'n$ intersections of f and ϕ.

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* $$\Omega_1^h = \frac{1}{m(m-1)\cdots(m-h+1)m'(m'-1)\cdots(m'-h+1)}\left(\frac{\partial^2}{\partial\lambda_1\partial\xi_2} - \frac{\partial^2}{\partial\lambda_2\partial\xi_1}\right)^h,$$

$$\Omega_2^k = \frac{1}{n(n-1)\cdots(n-k+1)n'(n'-1)\cdots(n'-k+1)}\left(\frac{\partial^2}{\partial\mu_1\partial\eta_2} - \frac{\partial^2}{\partial\mu_2\partial\eta_1}\right)^k.$$

The $(1\,,\ 1)$ transvectant is

$$(f\phi)_{11} = (ab)(a\beta)a_\lambda^{m'-1}b_\lambda^{m'-1}a_\mu^{n-1}\beta_\mu^{n'-1}$$
$$= f_{11}\phi_{22} - f_{12}\phi_{22} - f_{21}\phi_{12} + f_{22}\phi_{11}\,;$$

or, when the forms coincide,

$$(ff)_{11} = (aa')(aa')a_\lambda^{m-1}a_\lambda'^{m-1}a_\mu^{n-1}a_\mu'^{n-1}$$
$$= 2f_{11}f_{22} - 2f_{12}f_{21}\,.$$

For this case the first theorem of this section gives:

The $(1\,,\ 1)$ *transvectant of two curves is the locus of points whose polar circles with respect to the two curves are orthogonal.*

The $(1\,,\ 1)$ *transvectant of a curve over itself is the locus of points whose polar circles are degenerate. Therefore* $(ff)_{11} = 0$ *cuts* f *in its minimal points, the number of the latter being* $4mn - 2m - 2n$. By a minimal point of a curve is meant one where the tangent line is a minimal line. At such a point the polar circle consists of the pair of minimal lines through the point.

§14. *Some simultaneous covariants.*—Denote the polar circles of a point P with respect to the curves f, ϕ, $\psi \cdots$ by C_f, C_ϕ, $C_\psi \cdots$; so that

$$C_f = a_\xi^{m-1}a_\eta^{n-1}a_\lambda a_\mu = f_{11}\lambda_1\mu_1 + f_{12}\lambda_1\mu_2 + f_{21}\lambda_2\mu_1 + f_{22}\lambda_2\mu_2\,,\ \text{etc.}$$

Then, from § 6 and § 13,

(1)
$$I(C_f,\ C_f) = (ff)_{11}\,,\qquad I(C_fC_\phi) = (f\phi)_{11}\,,$$
$$K(C_fC_\phi C_\psi C_\chi) = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ \psi_{11} & \psi_{12} & \psi_{21} & \psi_{22} \\ \chi_{11} & \chi_{12} & \chi_{21} & \chi_{22} \end{vmatrix} = S_1\,.$$

Any point P may be considered as a degenerate circle, i. e., as the circle with the equation $\overline{P} = (\xi\lambda)(\eta\mu) = 0$; adding this to the system of polar circles, we have

$$I(C_f\overline{P}) = a_\xi a_\eta a_\xi^{m-1}a_\eta^{n-1} = f\,,$$
$$K(C_fC_\phi C_\psi\overline{P}) = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ \psi_{11} & \psi_{12} & \psi_{21} & \psi_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \end{vmatrix} = S_2\,.$$

From these formulæ we can write down at once any covariant which is defined by a property of the polar circles and $\bar{P}$. Thus the locus of points such that the polar circles with respect to f and ϕ are tangent, is $(ff)_{11}(\phi_1\phi)_{11} - (f\phi)^2_{11} = 0$ (§ 8). The locus defined by C_f, C_ϕ and $\bar{P}$ having a common point is

$$\begin{vmatrix} (ff)_{11} & (f\phi)_{11} & f \\ (\phi f)_{11} & (\phi\phi)_{11} & \phi \\ f & \phi & 0 \end{vmatrix} = 0 .$$

The defining property of S_1 above is that $C_f C_\phi C_\psi C_\chi$ have a common orthogonal circle. Symbolically,

$$S_1 = \{(ac)(bd)(\alpha\beta)(\gamma\delta) - (ab)(cd)(\alpha\gamma)(\beta\delta)\} a_\lambda^{m-1}\alpha_\mu^{n-1} \cdots d_\lambda^{m'''-1}\delta_\mu^{n'''-1} .$$

Its square is expressible in simple transvectants thus:

$$S_1^2 = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \times \begin{vmatrix} f_{22} & f_{21} & f_{12} & f_{11} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} (ff)_{11} & (f\phi)_{11} & (f\psi)_{11} & (f\chi)_{11} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \end{vmatrix} .$$

Similarly S_2 is the locus of points $\bar{P}$ for which $C_f C_\phi C_\chi$ and $\bar{P}$ have a common orthogonal circle ;

$$S_2 = \{(ac)(\alpha\beta)b_\lambda\gamma_\mu - (ab)(\alpha\gamma)c_\lambda\beta_\mu\} a_\lambda^{m-1}\alpha_\mu^{n-1} b_\lambda^{m'-1}\beta_\mu^{n'-1} c_\lambda^{m''-1}\gamma_\mu^{n''-1} ;$$

and

$$(2) \qquad S_2^2 = \begin{vmatrix} (ff)_{11} & (f\phi)_{11} & (f\psi)_{11} & f \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot \\ f & \phi & \psi & 0 \end{vmatrix} .$$

§ 15. *The discriminants and the nodal invariant.*—If in $f = a_\lambda^m \alpha_\mu^n$ we consider λ or μ as being the only variable, and take the corresponding discriminants, we obtain two simple binary forms $D_1(\lambda)$, $D_2(\mu)$, which may be called the *discriminants* * of f. Let the symbolic discriminant of α_μ^n be $\Delta_1(\alpha, \beta, \cdots)$; it will contain in all $2(n-1)$ symbols $\alpha, \beta, \cdots$. Similarly the discriminant $\Delta_2(a, b, \cdots)$ of a_λ^n contains $2(m-1)$ symbols ; D_1 and D_2 are then

$$(1) \qquad \begin{aligned} D_1(\lambda) &= \Delta_1(\alpha, \beta, \cdots) a_\lambda^m b_\lambda^m \cdots , \\ D_2(\mu) &= \Delta_2(a, b, \cdots) \alpha_\mu^n \beta_\mu^n \cdots , \end{aligned}$$

their orders being $2(n-1)m$ and $2(m-1)n$. They represent the tangent minimal lines of f, the total number of such lines being $4mn - 2m - 2n$ as found before (§ 13).

* CAPELLI, *Sopra la corrispondenza* (2, 2), Giornale di Matematiche, vol. 17, p. 75, 1879.

We consider now certain invariants of f which have simple geometric interpretations. Let $\delta_1 = 0$ be the condition for two λ-tangents following together; $P_1 = 0$ the condition for an inflexional λ-tangent; $Q_1 = 0$ the condition for a double λ-tangent; and $\delta_2 = 0$, $P_2 = 0$, $Q_2 = 0$ the corresponding conditions for μ-tangents.

From geometric considerations,

$$\delta_1 = N^a P_1^b Q_1^c, \quad \delta_2 = N^a P_2^b Q_2^c.$$

To determine a, b, c we must calculate the degrees of the invariants. The invariant δ_1 is the discriminant of $D_1(\lambda)$; its degree is therefore

$$4(n-1)\{2m(n-1)-1\}.$$

Again P_1, Q_1, equated to 0, are the conditions for the existence of a λ for which the equation $f(\lambda, \mu) = 0$ has a triple root, and two double roots respectively; their degrees may be found then, from SALMON's general formulæ for the order and weight of systems of equations.* It is thus found that the degree of P_1 is $6m(n-2)$, and the degree of Q_1 is $4m(n-2)(n-3)$.

To determine a, b, c, we have then the equations:

$$4(n-1)\{2m(n-1)-1\} = a[N] + 6bm(n-2) + 4cm(n-2)(n-3),$$

$$4(m-1)\{2n(m-1)-1\} = a[N] + 6bn(m-2) + 4cn(m-2)(m-3),$$

where $[N]$ denotes the degree of N. Equating the values of $a[N]$, we obtain $b = 3$, $c = 2$; and determining a from the values of the invariants for the circle, we have

$$(2) \qquad \delta_1 = NP_1^3 Q_1^2, \quad \delta_2 = NP_2^3 Q_2^2.$$

The nodal invariant is the greatest common divisor of the discriminants δ_1, δ_2, of the discriminants D_1, D_2, of f. Its degree is $6mn - 4(m+n-1)$.†

§ 16. *Contravariants.*—Besides invariants and covariants, forms involving the coördinates of a variable circle are also of geometric importance; such a form, equated to zero, denotes a complex of circles with invariant relations to the given curve or curves. The variable circle will be taken in the form:

$$(1) \qquad C = r_{11}\lambda_1\mu_1 + r_{12}\lambda_1\mu_2 + r_{21}\lambda_2\mu_1 + r_{22}\lambda_2\mu_2 = r_\lambda \rho_\mu.$$

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*SALMOND-FIELDER, *Raumgeometrie*, 3d edition, vol. II, p. 595, formulæ for cases c) and d).

† The intersection of two surfaces can have a node only if the surfaces are tangential. Therefore in the case where f is equi-ordinal, $f = a_\lambda^n a_\mu^n$, the problem of finding N is equivalent to the problem of finding the tact-invariant of a quadric and the general surface of order n. The above result then gives a method of finding this tact-invariant by purely binary considerations. The degree is $6n^2 - 8n + 4$, as may also be found from quaternary considerations (SALMON-FIELDER, *Raumgeometrie*, 3d edition, II, 609).

The circle C intersects $f = a_\lambda^m a_\mu^n$ in $m + n$ points, whose coördinates are obtained by the elimination of λ or μ from $C = 0$, $f = 0$; thus we obtain two binary forms:

$$(2)\qquad \begin{aligned} L(\lambda) &= (f,\ C^n)_{0n} = (a\rho') \cdots (a\rho^{(n)}) a_\lambda^m r'_\lambda \cdots r_\lambda^{(n)} = t_\lambda^{m+n}, \\ M(\mu) &= (f,\ C^m)_{m0} = (ar') \cdots (ar^{(m)}) a_\mu^n \rho'_\mu \cdots \rho_\mu^{(m)} = t_\mu^{m+n}, \end{aligned}$$

the first representing the λ-lines, and the second the μ-lines, through the intersections of C and f. That it is necessary in general to consider only one of these forms is shown by the theorem:

If C is not degenerate, the forms L and M are equivalent.

This is proved by showing that the substitution

$$(3)\qquad \begin{aligned} \lambda_1 &= \quad r_{21}\mu_1 + r_{22}\mu_2 = \quad r_2\rho_\mu, \\ \lambda_2 &= -r_{11}\mu_1 - r_{12}\mu_2 = -r_1\rho_\mu, \end{aligned}$$

whose modulus

$$-\tfrac{1}{2}(rr')(\rho\rho') = -\tfrac{1}{2}\sigma'$$

is not zero by assumption—transforms L into M except for a factor. We have

$$L' = (a\rho') \cdots (a\rho^{(n)})\bar{\bar{\rho}}'_\mu \cdots (a\bar{r}^{(m)})\bar{\rho}_\mu^{(m)}(r'\bar{\bar{r}}')\bar{\bar{\rho}}'_\mu \cdots (r^{(n)}\bar{\bar{r}}^{(n)})\bar{\bar{\rho}}_\mu^{(n)};$$

but

$$(r\bar{\bar{r}})(a\rho)\bar{\rho}_\mu = -\sigma' a_\mu;$$

therefore

$$L' = (-1)^n \sigma'^n M.$$

The equivalence may also be seen geometrically; for L and M on the circle C represent the same points in different systems of linear coördinates.

From L or M a large class of contravariants may be deduced, since every invariant of L or M is a contravariant of f. Thus *to every invariant of a binary form of order $m + n$, corresponds a contravariant of the double binary form $f = a_\lambda^m a_\mu^n$*; this contravariant represents the complex of circles, each of which cuts f in $m + n$ points having the property denoted by the vanishing of the binary invariant.*

As an application consider the discriminants D_L and D_M, of L and M. The discriminant D_L can vanish only if the circle C is tangent to f, or if two of the points of intersection lie on a λ-line—the latter case occurring only where C is degenerate. If we denote the complex of tangent circles by Φ, we have

$$D_L = \Phi^a \sigma'^b.$$

* Compare the *Clebsch Uebertragungsprincip*, LINDEMANN, *Geometrie*, vol. I, p. 276. Similarly every covariant of the binary form gives a mixed concomitant of f, i. e., a concomitant involving the coördinates of both a variable point and a variable circle.

Reasoning as in the preceding article, we find

$$D_L = \Phi\sigma'^{n(n-1)}, \quad D_M = \Phi\sigma'^{m(m-1)}. \tag{4}$$

The tangential complex Φ may therefore be obtained by forming either discriminant D_L or D_M, and neglecting the irrelevant complex of degenerate circles. Its degree (in coefficients of f) is $2(m+n-1)$, and its class (degree in coefficients of C) is $2mn$.*

Another method which is useful in the formation of contravariants, is founded upon a principle by which from any equi-covariant, i. e., a covariant which is equi-ordinal—we may obtain a corresponding contravariant; and conversely, from any contravariant, a corresponding covariant. Any contravariant Ω of a system of forms can be written

$$\Omega = (tr)(tr')\cdots(tr^{(k-1)})(\tau\rho)(\tau\rho')\cdots(\tau\rho^{(k-1)}). \tag{5}$$

If in (5) we put $(tr) = t_\lambda$, $(\tau\rho) = \tau_\mu$, we obtain a covariant

$$V = t_\lambda^k \tau_\mu^k, \tag{6}$$

involving λ and μ in the same orders; and similarly, from any equi-covariant (6) a corresponding contravariant (5) may be derived.

From any contravariant Ω, a covariant V may be obtained by substituting point coördinates λ, μ for the symbols r, ρ of the variable circle which occurs in Ω; and from any equi-covariant V, a contravariant Ω may be obtained by substituting for λ, μ the symbols of a circle.

Using the definition of the polar circle of a given circle with respect to a curve, which is given later (§ 24), the geometric relations of corresponding covariants and contravariants are:

$V = 0$ is the locus of the vertices of the degenerate circles contained in $\Omega = 0$; while $\Omega = 0$ is the complex of circles which are orthogonal to their polar circles with respect to the curve $V = 0$.

CHAPTER IV.

Connection of the Binary and Quaternary Theories.

In the preceding, the binary and quaternary methods have been considered independently of each other; but the fact that both are methods which apply to

* The class of the tangential complex may also be obtained from projective considerations. The class of Φ is the number of circles in any pencil of circles, which are tangent to f; it is therefore also the number of lines through a point, which are tangent to f, i. e., it is the (projective) class of the curve f, which from Plucker's formula is

$$(m+n)(m+n-1) - m(m-1) - n(n-1) = 2mn.$$

the same geometry, indicates a certain isomorphism between the two. The explicit form of this connection will be studied in the present section. It will be shown how the apolar surface which proved to be fundamental in the quaternary theory, also presents itself from the binary point of view; and how we may pass from a quaternary concomitant to a corresponding binary concomitant.

§ 17. *Parametric representation of the fundamental quadric.*—Consider any four circles not belonging to a linear system,

$$(1) \qquad C_1 = \bar{a}_\lambda \bar{a}_\mu, \quad C_2 = \bar{b}_\lambda \bar{\beta}_\mu, \quad C_3 = \bar{c}_\lambda \bar{\gamma}_\mu, \quad C_4 = \bar{d}_\lambda \bar{\delta}_\mu .$$

In connection with these there are the covariant orthogonal circles

$$(2) \qquad C_1' = \bar{\bar{a}}_\lambda \bar{\bar{a}}_\mu, \quad C_2' = \bar{\bar{b}}_\lambda \bar{\bar{\beta}}_\mu, \quad C_3' = \bar{\bar{c}}_\lambda \bar{\bar{\gamma}}_\mu, \quad C_4' = \bar{\bar{d}}_\lambda \bar{\bar{\delta}}_\mu ,$$

where for example C_1' is the circle orthogonal to $C_2 C_3 C_4$, so that

$$\bar{\bar{a}}_\lambda \bar{\bar{a}}_\mu = (\bar{b}\bar{d})(\bar{\beta}\bar{\gamma})\bar{c}_\lambda \bar{\delta}_\mu - (\bar{b}\bar{c})(\bar{\beta}\bar{\delta})\bar{d}_\lambda \bar{\gamma}_\mu , \quad \text{etc.;}$$

and the invariants

$$(3) \qquad \begin{aligned} I_{11} &= (\bar{a}\bar{a}')(\bar{a}\bar{a}'), \quad I_{12} = (\bar{a}\bar{b})(\bar{a}\bar{\beta}), \quad \text{etc.,} \\ I_{11}' &= (\bar{\bar{a}}\bar{\bar{a}}')(\bar{\bar{a}}\bar{\bar{a}}'), \quad I_{12}' = (\bar{\bar{a}}\bar{\bar{b}})(\bar{\bar{a}}\bar{\bar{\beta}}), \quad \text{etc.,} \end{aligned}$$

or

$$I_{ik}' = \text{minor of } I_{ik} \text{ in the determinant } |I_{ik}| ,$$

$$(4) \qquad D = K_{1234} = (\bar{a}\bar{c})(\bar{b}\bar{d})(\bar{a}\bar{\beta})(\bar{\gamma}\bar{\delta}) - (\bar{a}\bar{b})(\bar{c}\bar{d})(\bar{a}\bar{\gamma})(\bar{\beta}\bar{\delta}), \quad D^2 = |I_{ik}'| ,$$

$$D' = K_{1234}' = D^3, \quad D'^2 = |I_{ik}'| = D^6 .$$

Let $x_1 x_2 x_3 x_4$ be homogeneous coördinates in space, and put

$$(5) \qquad \rho x_1 = \bar{a}_\lambda \bar{a}_\mu, \quad \rho x_2 = \bar{b}_\lambda \bar{\beta}_\mu, \quad \rho x_3 = \bar{c}_\lambda \bar{\gamma}_\mu, \quad \rho x_4 = \bar{d}_\lambda \bar{\delta}_\mu .$$

This is the parametric representation of a quadric; for the four circles $C_1 C_2 C_3 C_4$ are connected by the identity (§7):

$$\begin{vmatrix} |I_{ik}|, & C_i \\ C_k, & 0 \end{vmatrix} \equiv 0 ,$$

where the left hand member denotes the fifth order determinant obtained by bordering the fourth order determinant $|I_{ik}|$.

The equation of the quadric (5) is therefore

$$(6) \qquad Q = p_x^2 = - \begin{vmatrix} |I_{ik}|, & x_i \\ x_k, & 0 \end{vmatrix} = \sum I_{ik}' x_i x_k = 0 .$$

This represents the most general non-degenerate quadric; for the invariants I'_{ik} are independent, and the discriminant

(7) $$\Delta = |I'_{ik}| = D^6 = \tfrac{1}{24}(pp'p''p''')^2,$$

by the assumption that $C_1 C_2 C_3 C_4$ are linearly independent, can not vanish. Again the x's may be regarded as the tetracyclic coördinates of the point λ, μ, the base circles being $C_1 C_2 C_3 C_4$, and the identity being $Q \equiv 0$.

The tangential equation of Q is,

(8) $$\bar{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2 = D^4 \sum I_{ik} u_i u_k = 0.$$

Any circle $C = r_\lambda \rho_\mu$ is connected with $C_1 C_2 C_3 C_4$ by the identity [(4) § 7],

(9) $$D r_\lambda \rho_\mu \equiv (r\bar{\bar{a}})(\rho\bar{\bar{a}})C_1 + (r\bar{\bar{b}})(\rho\bar{\bar{\beta}})C_2 + \text{etc.};$$

therefore its quaternary coördinates are

(10) $$u_1 = (\bar{\bar{a}}r)(\bar{\bar{a}}\rho), \quad u_2 = (\bar{\bar{b}}r)(\bar{\bar{\beta}}\rho), \quad u_3 = (\bar{\bar{c}}r)(\bar{\bar{\gamma}}\rho), \quad u_4 = (\bar{\bar{d}}r)(\bar{\bar{\delta}}\rho).$$

The formulæ (5) institute a (1, 1) correspondence between the points (λ, μ) of a plane and points x_i of the quadric (6). In this correspondence the points of a circle in the plane correspond to the points of a plane section of Q; the formulæ (10) show how the coördinates of corresponding circles and planes are connected.

§ 18. *The apolar surface corresponding to an equi-form.*—Consider any double binary form

(1) $$f = a_\lambda^n a_\mu^n$$

whose partial orders are equal. Such an "equi-form" may be expressed in terms of the four bilinear forms $C_1 C_2 C_3 C_4$. For by (9) § 17,

$$D a_\lambda a_\mu \equiv (a\bar{\bar{a}})(a\bar{\bar{a}})\bar{a}_\lambda \bar{a}_\mu + \text{etc.};$$

therefore,

(2) $$\begin{aligned} D^n a_\lambda^n a_\mu^n \equiv & \left[(a\bar{\bar{a}}')(a\bar{\bar{a}}')\bar{a}'_\lambda \bar{a}'_\mu + (a\bar{\bar{b}}')(a\bar{\bar{\beta}}')\bar{b}'_\lambda \bar{\beta}'_\mu + \cdots\right] \\ & \times \left[(a\bar{\bar{a}}'')(a\bar{\bar{a}}'')\bar{a}''_\lambda \bar{a}''_\mu + \cdots \quad \cdots \quad \cdots \quad \cdots\right] \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \times \left[(a\bar{\bar{a}}^{(n)})(a\bar{\bar{a}}^{(n)})\bar{a}_\lambda^{(n)} \bar{a}_\mu^{(n)} + \quad \cdots \quad \cdots \quad \cdots\right]. \end{aligned}$$

Introducing the x's by (1) § 17, we have

(3) $$\left\{ \begin{aligned} & D^n a_\lambda^n a_\mu^n = A_x^n, \\ & \text{where the symbols } A \text{ are defined by} \\ & A_1 = (a\bar{\bar{a}})(a\bar{\bar{a}}) = (a\bar{\bar{a}}')(a\bar{\bar{a}}') = \cdots, \quad A_2 = (\bar{\bar{a}}b)(\bar{\bar{a}}\beta) = (\bar{\bar{a}}b')(\bar{\bar{a}}\beta') = \cdots, \\ & A_3 = (a\bar{\bar{c}})(a\bar{\bar{\gamma}}) = (a\bar{\bar{c}}')(a\bar{\bar{\gamma}}') = \cdots, \quad A_4 = (a\bar{\bar{d}})(a\bar{\bar{\delta}}) = (a\bar{\bar{d}}')(a\bar{\bar{\delta}}') = \cdots. \end{aligned} \right.$$

Starting with the binary form $f = a_\lambda^n \boldsymbol{a}_\mu^n$, we obtain in this way a perfectly definite quaternary form $F = A_x^n$. For all points of the quadric Q, the form F vanishes when and only when f does—that is to say, in space, $F = 0$ represents a surface which cuts the quadric Q in the curve $f = 0$,—but obviously this alone does not determine F. The determination is completed in the

FUNDAMENTAL THEOREM.—*The quaternary form* $F = A_x^n$, *which corresponds* [by (3)] *to a double binary form* $f = a_\lambda^n \boldsymbol{a}_\mu^n$, *represents that surface through the curve* $f = 0$ *on the quadric* Q, *which is apolar to* Q.

To prove this, consider the covariant $A_P^2 A_x^{n-2}$:

$$A_P^2 = D^4 \sum I_{ik} A_i A_k = -\begin{vmatrix} |I'_{ik}| & A_i \\ A_k & 0 \end{vmatrix}$$

$$= \begin{vmatrix} |I'_{ik}| & (a\bar{\bar{a}})(\boldsymbol{a}\bar{\bar{\boldsymbol{a}}}) \\ (a\bar{\bar{a}}')(\boldsymbol{a}\bar{\bar{\boldsymbol{a}}}') & (aa)(\boldsymbol{a}\boldsymbol{a}) \end{vmatrix} + (aa)(\boldsymbol{a}\boldsymbol{a})\,|I'_{ik}| = 0,$$

by § 3; therefore $A_P^2 A_x^{n-2} \equiv 0$, and F is apolar to Q (§ 3).

This result gives a method for determining the apolar surface F passing through a curve on Q, which is essentially different from that in § 9. Let the curve be determined by any surface $F' = 0$ which passes through it; by the substitution (5) § 17, F' becomes a double binary form $f = a_\lambda^n \boldsymbol{a}_\mu^n$; then by (3) we pass from f to a quaternary form F which will be the required apolar form.

If Q is taken in the form

$$Q = 2x_1x_4 - 2x_2x_3,$$

the parametric representation may be put into the normal form

$$x_1 = \lambda_1\mu_1, \quad x_2 = \lambda_1\mu_2, \quad x_3 = \lambda_2\mu_1, \quad x_4 = \lambda_2\mu_2.$$

Then the F corresponding to $f = a_\lambda^n \boldsymbol{a}_\mu^n$ is

$$F = (a_1\boldsymbol{a}_1x_1 + a_1\boldsymbol{a}_2x_2 + a_2\boldsymbol{a}_1x_3 + a_2\boldsymbol{a}_2x_4)^n;$$

and the passage from F' to F may be formulated as follows: let

$$F' = \sum cCx_1^\alpha x_2^\beta x_3^\gamma x_4^\delta,$$

where c denotes the numerical and C the literal coefficients; substitute $x_1 = \lambda\mu$, $x_2 = \lambda$, $x_3 = \mu$, $x_4 = 1$, so that F' becomes, say,

$$\sum (c_1C_1 + c_2C_2 + \cdots)\lambda^r\mu^s;$$

in this put $C_1 = C_2 =$ etc.; the result will be F.

§ 19. *Coincidence of the binary and quaternary polar theories.*—It has been seen in chapter II that the study of a curve in tetracyclic coördinates depends intimately upon the apolar form F; and by means of the quaternary polars of F, a complete quaternary polar theory of curves might have been developed. This was not done however, only binary polars being considered in § 12. The question may then arise, whether curves have two distinct inversion polar theories, a binary and a quaternary.

The apolar form corresponding to

$$f = a_\lambda^n \alpha_\mu^n,$$

is, by (3),

$$F = A_x^n = [(a\bar{\bar{a}}')(\alpha\bar{\bar{\alpha}}')x_1 + \cdots]\,[(a\bar{\bar{a}}'')(\alpha\bar{\bar{\alpha}}'')x_1 + \cdots]\cdots[(a\bar{\bar{a}}^{(n)})(\alpha\bar{\bar{\alpha}}^{(n)})x_1 + \cdots].$$

For the points of Q, we have

$$A_x^n = D^n a_\lambda^n \alpha_\mu^n, \quad A_y = D a_\xi \alpha_\eta,$$

$$A_y^k A_x^{n-k} = D^n a_\xi^k \alpha_\eta^k a_\lambda^{n-k} \alpha_\mu^{n-k},$$

where y or ξ, η is any point of Q. Again, since F is apolar to Q, so are all its polars $A_y^k A_x^{n-k}$. Therefore

If f and F are corresponding forms, so are their polars $P_{kk}(f)$ and $P_k(F)$.

This gives a characteristic property of the apolar surface F. *The polar surfaces of any point of Q with respect to the surface F intersect Q in the (equi-ordinal) polars of the same point with respect to the curve f;* or briefly, *the two polar theories coincide.**

§20. *Symbolic relations.*† —The object of this section is to develop the formulæ by which, from a quaternary concomitant, we may pass to a corresponding binary concomitant and conversely. The fundamental quadric, its contravariant and its discriminant are

$$(1) \quad Q = p_x^2, \quad \overline{Q} = u_P^2 = \tfrac{1}{6}(pp'p''u)^2, \quad \Delta = \tfrac{1}{24}(pp'p''p''')^2 = \tfrac{1}{4}p_P^2 = D^6.$$

The symbols of binary forms, corresponding to quaternary forms whose symbols are $A, B, C\cdots$, will be written a, α; b, β; c, γ; $\cdots$, so that by (3) § 18,

$$(2) \quad \begin{aligned} A_1 &= (a\bar{\bar{a}})(\alpha\bar{\bar{\alpha}}), & A_2 &= (a\bar{\bar{b}})(\alpha\bar{\bar{\beta}}), & &\cdots, \\ B_1 &= (b\bar{\bar{a}})(\beta\bar{\bar{\alpha}}), & B_2 &= (b\bar{\bar{b}})(\beta\bar{\bar{\beta}}), & &\cdots, \\ C_1 &= (c\bar{\bar{a}})(\gamma\bar{\bar{\alpha}}), & C_2 &= (c\bar{\bar{b}})(\gamma\bar{\bar{\beta}}), & &\cdots, \quad \text{etc.} \end{aligned}$$

* Attention is restricted here to the equi-ordinal binary polars, since the others can not be represented by a single quaternary form; if we include all the polars, the quaternary theory forms merely a part of the binary theory.

† For relations connecting ternary and simple binary forms, Cf. LINDEMANN, *Sur une représentation géométrique des covariants des formes binaires*, Bulletin Société Mathématique de France, vol. 5, pp. 113-125, 1876, vol. 6, pp. 195-208, 1877; also Mathematische Annalen, vol. 23.

From these we have the fundamental relations:

(3) $$(ABCD) = \begin{vmatrix} (a\bar{\bar{a}})(\alpha\bar{\bar{\alpha}}), & (a\bar{\bar{b}})(\alpha\bar{\bar{\beta}}), & \cdots \\ (b\bar{\bar{a}})(\beta\bar{\bar{\alpha}}), & (b\bar{\bar{b}})(\beta\bar{\bar{\beta}}), & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_2\alpha_1 & a_2\alpha_2 \\ b_1\beta_1 & b_1\beta_2 & b_2\beta_1 & b_2\beta_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \times \begin{vmatrix} \bar{\bar{a}}_1\bar{\bar{\alpha}}_1, & \bar{\bar{a}}_1\bar{\bar{\alpha}}_2, & \bar{\bar{a}}_2\bar{\bar{\alpha}}_1 & \bar{\bar{a}}_2\bar{\bar{\alpha}}_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= D^3\{(ac)(bd)(\alpha\beta)(\gamma\delta) - (ab)(cd)(\alpha\gamma)(\beta\delta)\} \qquad [(4)\ \S\ 17];$$

(4) $$A_PB_P = \tfrac{1}{6}(pp'p''A)(pp'p''B) = D^4\sum I_{ik}A_iB_k$$

$$= -\begin{vmatrix} |I'_{ik}| & A_i \\ B_k & 0 \end{vmatrix} = -\begin{vmatrix} |I'_{ik}| & (a\bar{\bar{a}})(\alpha\bar{\bar{\alpha}}) \\ (b\bar{\bar{a}}')(\beta\bar{\bar{\alpha}}') & (ab)(\alpha\beta) \end{vmatrix} + (ab)(\alpha\beta)|I'_{ik}|$$

$$= D^6(ab)(\alpha\beta) \qquad [(5)\ \S\ 7 \text{ and } (4)\ \S\ 17];$$

(5) $$\begin{cases} (pp'AB)(pp'CD) = 2D^6\begin{vmatrix} (ac)(\alpha\gamma) & (ad)(\alpha\delta) \\ (bc)(\beta\gamma) & (bd)(\beta\delta) \end{vmatrix}, \\ (pABC)(p'DEF) = -D^6\begin{vmatrix} (ad)(\alpha\delta), & (ae)(\alpha\epsilon), & (af)(\alpha\phi) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}, \\ (ABCD)(A'B'C'D') = D^6\begin{vmatrix} (aa')(\alpha\alpha'), & (ab')(\alpha\beta'), & (ac')(\alpha\gamma'), & (ad')(\alpha\delta') \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \\ \quad = D^6\{(ac)(bd)(\alpha\beta)(\gamma\delta) - (ab)(cd)(\alpha\gamma)(\beta\delta)\}\{(a'c')(b'd')(\alpha'\beta')(\gamma'\delta') \\ \qquad - (a'b_,)(c'd')(\alpha'\gamma')(\beta'\delta')\}; \end{cases}$$

(6) $$A_x = Da_\lambda\alpha_\mu;$$

$$(pABC)p_x = \tfrac{1}{2}\begin{vmatrix} \frac{\partial Q}{\partial x_1} & \frac{\partial Q}{\partial x_2} & \frac{\partial Q}{\partial x_3} & \frac{\partial Q}{\partial x_4} \\ A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \end{vmatrix};$$

and on substituting

$$\frac{1}{2}\frac{\partial Q}{\partial x_1} = I'_{11}x_1 + \cdots + I'_{14}x_4 = \begin{vmatrix} \bar{a}_\lambda \bar{a}_\mu & \bar{b}_\lambda \bar{\beta}_\mu & \cdots \\ I_{21} & I_{22} & \cdots \\ I_{31} & I_{32} & \cdots \\ I_{41} & I_{42} & \cdots \end{vmatrix} = DC'_1, \text{ etc.} \qquad [(2)\,\S\,17],$$

we obtain

$$(7) \qquad (pABC)p_x = D \begin{vmatrix} C'_1 & C'_2 & \cdots \\ A_1 & A_2 & \cdots \\ B_1 & B_2 & \cdots \\ C_1 & C_2 & \cdots \end{vmatrix} = D^4\{(ac)(a\beta)b_\lambda\gamma_\mu - (ab)(a\gamma)c_\lambda\gamma_\mu\}.$$

§ 21. *Transformation of the concomitants.*—The general G concomitant of a system of quaternary forms can be written (§ 1)

$$S = \sum \phi(p) T,$$

where T represents the general projective concomitant of the quaternary forms and the fundamental quadric. We may therefore apply the Clebsch-Aronhold symbolism: every G concomitant of the form A_x^n is symbolically a concomitant of the linear forms $A_x, B_x, \cdots$. From § 5, then, S can be symbolically expressed in terms of the types

$$(1) \qquad \begin{array}{ll} O: & A_x,\ A_PB_P,\ A_Pu_P; \\ O': & (ABCD),\ (ABCp)p_x,\ (ABCu). \end{array}$$

Furthermore the square of the type O' can be expressed in terms of the type O. Therefore S can be written

$$(2) \qquad S = R(O) + \sum O'R'(O),$$

where R and R' denote rational integral functions.

Every G' concomitant can be reduced to one of the forms

$$(3) \qquad R(O),\ \sum O'R'(O),$$

according as it is even or odd (§ 1).

Consider now the symbolic representation of the concomitants of double binary forms. Every concomitant of $a_\lambda^m a_\mu^n$ is expressible symbolically through the types: *

$$(ab),\ (a\beta),\ a_\lambda,\ a_\mu,\ (ar),\ (a\rho),$$

* If λ and μ are cogredient we must add the types (aa), a_μ, a_λ, $(a\rho)$, (ar); for distinct transformations of λ and μ, however, such types are not invariant. Cf. CAPELLI, Giornale di Matematiche, 1879, p. 71.

by the general theory of the Clebsch-Aronhold symbolism. If we limit ourselves to equi-forms $f = a_\lambda^n a_\mu^n$, every equi-concomitant B is symbolically a concomitant of the bilinear forms $a_\lambda a_\mu$, $b_\lambda b_\mu$, $\cdots$; and therefore (§ 6) expressible through the types

$$
(4)\quad
\begin{aligned}
\Omega: \qquad & [ab] = (ab)(\alpha\beta), \ [au] = (ar)(\alpha\rho), \ \{a\lambda\} = a_\lambda \alpha_\mu ; \\
& [abcd] = \{(ac)(bd)(\alpha\beta)(\gamma\delta) - (ab)(cd)(\alpha\gamma)(\beta\delta)\}, \\
\Omega': \qquad & [abcu] = \{(ac)(\alpha\beta)(br)(\gamma\rho) - (ab)(\alpha\gamma)(cr)(\beta\rho)\}, \\
& \{abc\lambda\} = \{(ac)(\alpha\beta)\, b_\lambda \gamma_\rho - (ab)(\alpha\gamma)\, c_\lambda \beta_\rho\}.
\end{aligned}
$$

Of the second type Ω', only the first power is necessary (§ 7); therefore

$$
(5)\qquad B = R(\Omega) + \textstyle\sum \Omega' R'(\Omega).
$$

Those B's which are invariant also for the transformations H, that is, concomitants of the group G', can be reduced to one of the forms:

$$
(6)\qquad R(\Omega), \quad \textstyle\sum' \Omega' R'(\Omega),
$$

*the first characterising those of even, and the second those of odd character.**

If now the forms $f = a_\lambda^n a_\mu^n$ and $F = F_x^n$ *correspond* in the sense of § 18, the types Ω, Ω' and O, O' are connected by the relations given in the preceding section. From (2), (3), (4), (5) we have then the fundamental

THEOREM.—*From every equi-concomitant of a system of double binary equi-forms, may be derived a concomitant of the system of corresponding (apolar) quaternary forms, by substituting for every type symbol Ω, Ω' a corresponding type symbol O, O'; conversely, a binary concomitant may be derived from every quaternary concomitant. In this transformation an even G' concomitant remains an even G' concomitant, and an odd remains odd. Geometrically, corresponding concomitants are equivalent, their vanishing having the same interpretation. The equivalence holds also in an algebraic sense, since corresponding forms are connected by the same syzygies, and a complete system remains complete after transformation.*

Another set of relations is obtained by considering, instead of the G and G' concomitants of the quaternary forms F, F', $\cdots$, the ordinary (projective) concomitants of the enlarged system: †

$$
Q, \ F, \ F', \ \cdots.
$$

* By the method employed in § 1, it may be shown that for G concomitants the factor produced by transformation is of the form $D_1^k D_2^l$, where $D_1 = (ad - bc)$, $D_2 = (\alpha\delta - \beta\gamma)$, (4) § 10; and for G' concomitants the factor produced by the transformations G is of the form $D_1^k D_2^l$, while that produced by the transformations H may be either $D_1^k D_2^l$ or $-D_1^k D_2^l$, thus creating the distinction of even and odd G' concomitants.

† Here, and in the following relations, the coefficients p_{ik} of the fundamental quadric Q are *not* regarded as numbers, i. e., they are not included in the domain from which the coefficients of concomitants may be chosen, as was the case in the preceding.

From § 5 it may be shown that every such concomitant T, multiplied by a sufficiently high power of Δ, can be symbolically expressed in one and only one of the forms

$$(7) \qquad R(\Delta, O), \quad \sum O'R'(\Delta, O),$$

the first form characterizing those of even, and the second those of odd weight. Employing the same symbolic transformation as in the preceding theorem, and comparing with (6), we obtain this result:

The equi-ordinal G' concomitants of a system of equi-forms

$$f = a_\lambda^n \alpha_\mu^n, \quad f' = b_\lambda^m \beta_\mu^m, \quad \cdots,$$

may be obtained from the ordinary concomitants of the fundamental quadric Q, and the corresponding apolar quaternary forms

$$F = A_x^n, \quad F' = B_x^m, \quad \cdots.$$

The converse is also true.

Every G concomitant is the sum of two G' concomitants; for denoting by $\overline{G}$ the result of operating on the concomitant G by means of an improper transformation,* we have

$$G = \frac{G + \overline{G}}{2} + \frac{G - \overline{G}}{2}.$$

Therefore from every equi-concomitant of the binary forms, we can obtain in general two concomitants T. Complete systems stand in this relation:

The binary concomitants corresponding to a complete T system form a complete equi-binary system. The T concomitants obtained from a complete equi-binary system, together with the discriminant Δ, form a complete T system, provided Δ^{-1} is regarded as an integral invariant.

§ 22. *Examples of the transformation of concomitants.*—The Hessian of

$$F = A_x^n$$

is

$$(ABCD)^2 A_x^{n-2};$$

if we apply (3) and (6) of § 20, and the symbols defined in (1) § 21, this becomes (omitting powers of D)

$$(1) \qquad [abcd]^2 \{a\lambda\}^{n-2},$$

which is the corresponding binary form. Expanded,

* We may derive $\overline{G}$ from G by interchanging, in the symbolic representation of G, a with α and λ with μ.

$$(2) \quad [abcd]^2 = \begin{vmatrix} (ab)(\alpha\beta), & (ac)(\alpha\gamma), & (ad)(\alpha\delta) \\ . & . & . \end{vmatrix}$$

$$= 2\{(ab)^2(cd)^2(\alpha\beta)^2(\gamma\delta)^2 - (ab)(bc)(cd)(da)(\alpha\beta)(\beta\gamma)(\gamma\delta)(\delta\alpha)\},$$

$$\{a\lambda\}^{n-2} = a_\lambda^{n-2}\alpha_\mu^{n-2}.$$

The Jacobian of four surfaces A_x^n, $B_x^{n_1}$, $C_x^{n_2}$, $D_x^{n_3}$,

$$J = (ABCD)A_x^{n-1}B_x^{n_1-1}C_x^{n_2-1}D_x^{n_3-1},$$

becomes
$$J' = [abcd]\{a\lambda\}^{n-1}\{b\lambda\}^{n_1-1}\{c\lambda\}^{n_2-1}\{d\lambda\}^{n_3-1},$$

which coincides with the covariant S_1 of § 14. The non-symbolic expressions are therefore

$$J = \begin{vmatrix} F_1 & F_2 & F_3 & F_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ . & . & . & . \\ . & . & . & . \end{vmatrix}, \quad J' = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ . & . & . & . \\ . & . & . & . \end{vmatrix}.$$

If one of the surfaces is the fundamental quadric Q, the Jacobian becomes $(ABCp)A_x^{n-1}B_x^{n_1-1}C_x^{n_2-1}p_x$, with the binary equivalent:

$$\{abc\lambda\}\{a\lambda\}^{n-1}\{b\lambda\}^{n_1-1}\{c\lambda\}^{n_2-1} = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ \phi_{11} & \phi_{12} & \phi_{21} & \phi_{22} \\ x_{11} & x_{12} & x_{21} & x_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \end{vmatrix},$$

which is the S_2 of § 14.

Consider now, as an example of the second kind, the problem of expressing binary transvectants in quaternary form. Let the binary forms be

$$f = a_\lambda^m\alpha_\mu^n, \quad \phi = b_\lambda^n\beta_\mu^n,$$

and the corresponding quaternary,

$$F = A_x^m, \quad \phi = B_x^n.$$

The transvectant is

$$(f\phi)_{kk} = (ab)^k(\alpha\beta)^k a_\lambda^{m-k}b_\lambda^{n-k}\alpha_\mu^{m-k}\beta_\mu^{n-k}.$$

Therefore, applying (4) and (6) of § 20,

$$(3) \qquad D^{m+n+4k}(f\phi)_{kk} = A_pA_{p'}\cdots A_{p^{(k-1)}}B_pB_{p'}\cdots B_{p^{(k-1)}}A_x^{m-k}B_x^{n-k}.$$

§ 23. *The osculating circle in invariant form.*—As an application of the preceding methods, consider the problem of determining the osculating circle at a point of the curve $f \equiv a_\lambda^n a_\mu^n = 0$. Let the corresponding quaternary form be $F = A_x^n$; then by applying HESSE's formula * for the osculating plane to the space curve

$$F \equiv A_x^n = 0, \quad Q \equiv p_x^2 = 0,$$

the osculating circle C' at the point y_i is found to be

$$(1) \qquad C' \equiv G(y) \sum F_i(y) x_i - H(y) \sum Q_i(y) x_i = 0,$$

where G and H are the covariants

$$G = \begin{vmatrix} |Q_{ik}| & F_i \\ F_k & 0 \end{vmatrix}, \quad H = \frac{1}{(n-1)^2} \begin{vmatrix} |F_{ik}| & Q_i \\ Q_k & 0 \end{vmatrix}.$$

To obtain the corresponding binary forms we must write these symbolically, the results being,

$$G = A_P B_P A_x^{n-1} B_x^{n-1},$$

$$H = \frac{1}{6(n-1)^2} (pABC)(p'ABC) p_x p'_x A_x^{n-2} B_x^{n-2} C_x^{n-2};$$

the binary forms are therefore,

$$g = (ff)_{11},$$

$$h = \frac{1}{6(n-1)^2} \{(ac)(\alpha\beta) b_\lambda \gamma_\mu - (ab)(\alpha\gamma) c_\lambda \beta_\mu\}^2 a_\lambda^{n-2} b_\lambda^{n-2} c_\lambda^{n-2} \alpha_\mu^{n-2} \beta_\mu^{n-2} \gamma_\mu^{n-2}.$$

The osculating circle at the point $P = \xi, \eta$ is

$$(2) \qquad C' \equiv g(\xi\eta) a_\xi^{n-1} \alpha_\eta^{n-1} a_\lambda \alpha_\mu - h(\xi\eta)(\xi\lambda)(\eta\mu) = 0.$$

This can be put into more convenient form by introducing the degenerate circle at P,

$$\bar{P} \equiv (\xi\lambda)(\eta\mu) = 0,$$

and the polar circle at P,

$$C \equiv a_\xi^{n-1} \alpha_\eta^{n-1} a_\lambda \alpha_\mu = 0;$$

equation (2) may then be written (omitting the arguments $\xi\eta$ in g and h),

$$(3) \qquad C' = gC - h\bar{P}.$$

---

* HESSE, Crelle, vol. 41, p. 283; or SALMON-FIEDLER, *Raumgeometrie*, 3d edition, vol. II, page 156, line 1.

The invariant of C' is

$$I(C'C') = g^2 I(CC) - 2gh I(C\bar{P}) + h^2 I(\bar{P}\bar{P});$$

but

$$I(CC) = (ff)_{11} = g, \quad I(C\bar{P}) = f, \quad I(\bar{P}\bar{P}) = 0;$$

therefore

$$I(C'C') = g(g^2 - fh) = g^3,$$

since only the points ξ, η on the curve f are considered.

The osculating circle is degenerate only at the points where g *cuts* f, i. e., *at the minimal points of* f (§ 13).

If $h = 0$, C' coincides with C; if they coincide, either $h = 0$ or C is degenerate. Therefore:

The points of f *where the osculating and polar circles coincide without degenerating, are cut out by the curve* h; *their number is* $2n(3n - 4)$.

From the orders of g and h in ξ, η we have:

Through any point of the plane, pass $6n(n - 1)$ *osculating circles of* f.

§ 24. *The apolar* relation of curves.*—Consider any two $C_{n,n}$'s:

$$f \equiv a_\lambda^n \alpha_\mu^n = 0, \quad f' \equiv b_\lambda^n \beta_\mu^n = 0.$$

Two C_{nn}*'s are said to be harmonic, conjugate, or apolar when their bilinear invariant vanishes.*

The bilinear invariant is (§ 13)

(1) $$(ff')_{nn} = (ab)^n (\alpha\beta)^n = \sum_{h,k} (-1)^{h+k} \binom{n}{h} \binom{n}{k} a_{hk} b_{n-h,\,n-k}.$$

Let the corresponding surfaces be

(2) $$F \equiv A_x^n = 0, \quad F' \equiv B_x^n = 0,$$

and their reciprocals with respect to Q,

(3) $$\phi \equiv u_\rho^n = 0, \quad \phi' \equiv u_\sigma^n = 0.$$

The polar of the plane u_i has the coördinates $u_P P_i$; therefore ϕ and ϕ' may be written

$$\phi = A_P A_{P'} \cdots A_{P^{(n-1)}} u_P \cdots u_{P^{(n-1)}} = u_\rho^n,$$

$$\phi' = B_P B_{P'} \cdots B_{P^{(n-1)}} u_P \cdots u_{P^{(n-1)}} = u_\sigma^n;$$

from which

$$\rho_i = A_P P_i, \quad \sigma_i = B_P P_i.$$

* The apolarity considered relates of course to the geometries (A) or (B) of the introduction, and is distinct from ordinary (projective) apolarity: two plane curves which are inversionally apolar are not in general projectively apolar; and conversely, projectively apolar curves are not in general inversionally apolar.

But from (3) § 22,

$$(ff')_{nn} = A_P \cdots A_{P^{(n-1)}} B_P \cdots B_{P^{(n-1)}};$$

substituting the values of either ρ_i or σ_i, we have

$$(4) \qquad (ff')_{nn} = (ab)^n(a\beta)^n = A^n_\sigma = B^n_\rho.$$

The invariant A^n_σ, however, is the bilinear invariant of the surfaces F and ϕ', and B^n_ρ is the bilinear invariant of F' and ϕ. Therefore,

*If C_{nn} and C'_{nn} are harmonic, so are the surface corresponding to either, and the reciprocal with respect to Q of the surface corresponding to the other.** [Added Sept. 28, 1900. More generally, *every* surface of the nth order through the one curve, is apolar to the reciprocal of the surface corresponding to the order.]

An interpretation of the apolar relation of C_{nn} and C'_{nn} can also be obtained without passing to space. With respect to the curve $f = 0$, every circle $r_\lambda \rho_\mu$ has a polar circle $(ar)(a\rho) \cdots (ar^{(n-2)})(a\rho^{(n-2)}) a_\lambda a_\mu = 0$; from the binary form of ϕ,

$$(5) \qquad \phi = (ar)(a\rho) \cdots (ar^{(n-1)})(a\rho^{(n-1)}),$$

we see then that ϕ can be defined as the complex of circles which are orthogonal to their polar circles with respect to f. Similarly ϕ' is the complex of circles which are orthogonal to their polar circles with respect to f'. The curves f and ϕ stand in the following reciprocal relation: f is the locus of the points contained in ϕ, and ϕ is the complex of circles which are orthogonal to their polar circles with respect to f. If f and f' are harmonic, so are the forms F and ϕ', and F' and ϕ.

If f and f' are cyclics, the bilinear invariant is

$$(6) \qquad \begin{aligned}(ff')_{22} = a_{00}b_{22} - 2a_{01}b_{21} + a_{02}b_{20} - 2a_{10}b_{12} + 4a_{11}b_{11} \\ + a_{22}b_{00} - 2a_{21}b_{01} + a_{20}b_{02} - 2a_{12}b_{10}.\end{aligned}$$

When the cyclics f and f' are apolar, the quadric ϕ is inscribed in an infinite number of tetrahedra which are self-polar with respect to F' and a similar relation exists between ϕ' and F.

In the plane the relation may be given by introducing the notion of circles which are conjugate with respect to a cyclic. The polar circle of any circle C with respect to f, is the locus of points whose polar circles are orthogonal to C; two circles are conjugate when each is orthogonal to the polar circle of the other. If then we call a set of four circles which are mutually conjugate a self-conjugate set we have:

If two cyclics f, f' are apolar, the complex ϕ of circles orthogonal to their polar circles with respect to f, contains an infinity of sets which are self-conjugate with

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* A corresponding interpretation of the conjugacy of two *simple* binary forms is given by SCHLESINGER, *Ueber conjugirte binäre Formen und deren geometrische Construction*, Mathematische Annalen, vol. 22, pp. 520–568, 1883.

respect to f'; and similarly with ϕ' and f. Conversely, if ϕ and f' or ϕ' and f stand in this relation, f and f' are apolar.

The importance of this theorem lies in the fact that it will serve as a basis for a synthetic theory of linear systems of cyclics, similar to that which has been developed for conics :* for a linear ν-membered system of cyclics,

$$t_1 f_1 + t_2 f_2 + \cdots + t_\nu f_\nu = 0,$$

may be defined as the totality of cyclics which are apolar to $9 - \nu$ linearly independent fixed cyclics.

§ 25. *Quaternary methods for the general double binary form.*—For the quaternary study of forms

$$f = a_\lambda^{n+\kappa} a_\mu^n,$$

whose partial orders are different, we may pass from f to an equi-concomitant. Thus we may consider

$$(ff)_{\kappa 0} = (ab)^\kappa a_\lambda^n a_\mu^n .$$

The concomitants of this form may be studied by quaternary methods ; but they will not, of course, include all the concomitants of f.

A system of equi-forms which is equivalent to f is given by any one of the systems of polars :

$$P_0 = a_\xi^\kappa a_\lambda^n a_\mu^n \quad \text{or} \quad P_h = a_\xi^{\kappa+h} a_\eta^h a_\lambda^{n-h} a_\mu^{n-h} \quad (h = 1, 2, \cdots, n-1).$$

A quaternary method of treating f consists then in the study of the $(1, 1)$ correspondence, defined by $P_h = 0$, between the points $(\xi\eta)$ of Q, and the two-parameter (non-linear) system of surfaces of order $n - h$, which corresponds, by § 18, to $P_h = 0$; or, in the study of the $(1, 1)$ correspondence, defined by $P_0 = 0$, between the generators $\lambda = \xi$ of one series on the quadric Q, and the one-parameter system of surfaces of order n which corresponds to $P_0 = 0$.

CHAPTER V.

The Cyclic Curves.

In tetracyclic coördinates the cyclic is represented by the quaternary quadric form a_x^2, and in minimal coördinates by the quadri-quadric double binary form $a_\lambda^2 a_\mu^2$. The first representation brings the inversion theory of the cyclic into relation with the theory of two quadric surfaces, while the second connects it with the general (2, 2) correspondence. The relation of the two points of view is most easily grasped by passing from the plane to the isomorphic problem in

* DARBOUX, Bulletin des Sciences Mathématiques, vol. 1. REYE, *Ueber Systeme und Gewebe von Kegelschnitten*, Crelle, vol. 82, pp. 54–83, 1877.

space of a bi-quadratic curve of the first species on a fixed quadric Q; for such a curve may be determined either by cutting Q with another quadric or by establishing a $(2, 2)$ correspondence between the two systems of generators on Q. The previous methods will now be applied to the discussion, from both points of view, of the concomitants of the cyclic—their complete systems, relations and geometric interpretations.*

§ 26. *System of two quadrics.*—Let the identity be $U' \equiv \sum p_{ik}x_ix_k \equiv p_x^2 = 0$, and the equation of the cyclic

$$U = \sum a_{ik}x_ix_k = a_x^2 = 0\,;$$

then (§ 4) the concomitants of the curve may be derived from the projective concomitants of the two quaternary quadrics U', U. Denoting the corresponding class quadrics by

$$\sigma' = u_P^2\,,\ \sigma = u_{\bar a}^2\,,$$

the invariants are

$$\Delta' = \tfrac{1}{24}(pp'p''p''')^2 = \tfrac{1}{4}p_P^2, \quad \Delta = \tfrac{1}{24}(abcd)^2 = \tfrac{1}{4}a_{\bar a}^2\,,$$

$$\theta' = \tfrac{1}{6}(pp'p''a)^2 = a_P^2, \quad \theta = \tfrac{1}{6}(pabc)^2 = p_{\bar a}^2\,,$$

$$\phi = \tfrac{1}{4}(pp'ab)^2\,;$$

the covariants are

$$U = a_x^2\,, \qquad U' = p_x^2\,,$$

$$S' = a_P b_P a_x b_x\,, \qquad S = p_{\bar a}p'_{\bar a}p_x p'_x\,,$$

$$Y' = \tfrac{1}{4}(abpp')b_P c_P p'_{\bar a}p''_{\bar a}a_x c_x p_x p''_x\,;$$

and the contravariants are

$$\sigma = u_{\bar a}^2 = \tfrac{1}{6}(abcu)^2\,, \quad \sigma' = u_P^2 = \tfrac{1}{6}(pp'p'''u)^2\,,$$

$$\tau' = \tfrac{1}{2}(pp'au)^2\,, \qquad \tau = \tfrac{1}{2}(pabu)^2\,,$$

$$\omega = \tfrac{1}{4}u_P u_{\bar a}(pp'au)(p''bcu)\begin{vmatrix} p_{\bar a} & p'_{\bar a} & a_{\bar a} \\ p_P & p'_P & a_P \\ (pp''bc) & (p'p''bc) & (ap''bc) \end{vmatrix}.$$

* A non-invariant discussion of these cyclic curves or bi-circular quartics is given in DARBOUX, *Sur une classe remarquable des courbes*, etc., Paris, 1872, reprint 1896.

In terms of these fifteen forms : *

$$(\Sigma_1)\qquad \begin{array}{ccccc} \Delta, & \Delta', & \theta, & \theta', & \phi, \\ U, & U', & S, & S', & Y', \\ \sigma, & \sigma', & \tau, & \tau', & \omega, \end{array}$$

every concomitant of U, U' involving at most one series of variables, can be rationally and integrally expressed.

Referred to their common self-polar tetrahedron, the quadrics may be written

$$U' = \sum x_i^2, \quad U = \sum a_i x_i^2;$$

and the complete system $(\sum_1)$ takes the normal form :

$$(1)\quad \begin{aligned} &\Delta' = 1, \quad \theta' = \sum a_i, \quad \phi = \sum a_i a_k, \quad \theta = \sum a_i a_k a_l, \quad \Delta = a_1 a_2 a_3 a_4; \\ &U' = \sum x_i^2, \quad U = \sum a_i x_i^2, \quad S' = \sum a_i^2 x_i^2, \quad S = \sum a_k a_l a_m x_i^2, \\ &Y' = (a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_3 - a_4)(a_4 - a_1)(a_2 - a_3) x_1 x_2 x_3 x_4; \\ &\sigma' = \sum u_i^2, \quad \tau' = \sum (a_k + a_l + a_m) u_i^2, \\ &\tau = \sum (a_k a_l + a_l a_m + a_m a_k) u_i^2, \quad \sigma = \sum a_k a_l a_m u_i^2, \\ &\omega = (a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_3 - a_4)(a_4 - a_1)(a_2 - a_3) u_1 u_2 u_3 u_4. \end{aligned}$$

§27. *Quaternary system of the cyclic.*—According to the method of §4, to apply the above results to the cyclic, we must replace U by that member F, of the pencil $U + kU'$, which is apolar to U'; this apolar quadric is [(6) or (7), §3]

$$(1)\qquad F = A_x^2 = \Delta' U - \frac{\theta'}{4} U'.$$

The complete quaternary system of the curve consists then of the "identical" form

$$\sigma' = u_P^2$$

(so called since it is the same for all curves), *and of the proper concomitants :* †

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* A non-symbolic discussion of these forms is given in SALMON-FIEDLER, *Raumgeometrie*, vol. I, Chapter XI. I have not thought it necessary to give the symbolic calculations by which the above results are obtained, as they are similar to those which occur in the corresponding ternary problem ; cf. GORDAN'S system, CLEBSCH-LINDEMANN, *Geometrie*, vol. I, p. 291. The completeness of (Σ_1) may be proved by comparing with MERTEN'S equivalent system, Wiener Berichte, 1889, 2 A, pp. 733–8.

† There is no invariant corresponding to θ' of Σ_1, since in the present case $A_P^2 = 0$.

$$I = \tfrac{1}{4}(pp'AB)^2, \quad J = \tfrac{1}{6}(pABC)^2, \quad k = \tfrac{1}{24}(ABCD)^2;$$

$$F = A_x^2, \quad G = A_P B_P A_x B_x, \quad H = p_{\bar{A}} p'_{\bar{A}} p_x p'_x,$$

$$(2) \qquad Y = \tfrac{1}{4}(ABpp')B_P C_P p'_{\bar{A}} p''_{\bar{A}} A_x C_x p_x p''_x;$$

$$\phi = \tfrac{1}{2}(pp'Au)^2, \quad \psi = \tfrac{1}{2}(pABu)^2, \quad \chi = u_{\bar{A}}^2,$$

$$Z = \tfrac{1}{4} u_P u_{\bar{A}} (pp'Au)(p''BCu) \begin{vmatrix} p_{\bar{A}} & p'_{\bar{A}} & A_{\bar{A}} \\ p_P & p'_P & A_P \\ (pp''BC) & (p'p''BC) & (Ap''BC) \end{vmatrix}.$$

These concomitants:

$$(\Sigma_2) \qquad \begin{matrix} I, & J, & K, & & \\ F, & G, & H, & Y, & \\ \phi, & \psi, & \chi, & Z, & \sigma' \end{matrix}$$

are also, by (1), concomitants of U' and U; it follows that $(\sum_2)$ is expressible integrally through $(\sum_1)$. To obtain the explicit relations, it is sufficient to consider both systems $(\sum_1)$ and $(\sum_2)$ in their normal forms; the normal form of $\sum_1$ has been given in § 26, and similarly the normal form of $\sum_2$ (omitting the identical form) is

$$\begin{aligned}
& I = \sum \beta_i \beta_k, \quad J = \sum \beta_i \beta_k \beta_l, \qquad K = \beta_1 \beta_2 \beta_3 \beta_4, \\
& F = \sum \beta_i x_i^2, \quad G = \sum \beta_i^2 x_i^2, \qquad H = \sum \beta_k \beta_l \beta_m x_i^2, \\
(3) \qquad & Y = (\beta_1 - \beta_2) \cdots (\beta_3 - \beta_4) x_1 x_2 x_3 x_4; \\
& \phi = -\sum \beta_i u_i^2, \quad \psi = \sum (I + \beta_i^2) u_i^2, \quad \chi = \sum \beta_k \beta_l \beta_m u_i^2, \\
& Z = (\beta_1 - \beta_2) \cdots (\beta_3 - \beta_4) u_1 u_2 u_3 u_4.
\end{aligned}$$

The a's and β's are connected, from (1), by the relation:

$$\beta_i = \Delta' a_i - \frac{\theta'}{4},$$

so that $\sum \beta_i = 0$. Introducing the value of β_i in the above normal forms, and comparing the results with (1) § 26, we obtain, after some calculation, the required relations between $\sum_1$ and $\sum_2$ as follows:

$$\begin{aligned}
8I &= 8\Delta'^2\phi - 3\Delta'\theta'^2, \\
16J &= 16\Delta'^3\theta - 8\Delta'^2\theta'\phi + 2\Delta'\theta'^3, \\
(4) \qquad 256K &= 256\Delta'^4\Delta - 64\Delta'^3\theta'\theta - 16\Delta'^2\theta'^{12}\phi - 3\Delta'\theta'^4, \\
4F &= 4\Delta' U - \theta' U', \\
16G &= 16\Delta'^2 S' - 8\Delta'\theta' U + \theta'^2 U',
\end{aligned}$$

$$
(4)\quad
\begin{aligned}
64H &= 64\Delta'^3 S + 12\Delta'\theta'^2 U - 16\Delta'^2\theta' S' + (3\theta'^3 - 16\theta.\phi)U',\\
Y &= \Delta'^4 Y',\\
4\phi &= 4\Delta'\tau' - 3\theta'\sigma',\\
16\psi &= 16\Delta'^2\tau - 8\Delta'\theta'\tau' + 3\theta'^2\sigma',\\
64\chi &= 64\Delta'^3\sigma - \Delta'^2\theta'\tau + 4\Delta'\theta'^2\tau' + \theta'^3\sigma',\\
Z &= \Delta'^4\omega.
\end{aligned}
$$

The formulæ (2) give the concomitants of the cyclic in terms of the identity $U' = 0$ and the apolar form F, while the formulæ (4) give them in terms of the identity $U' = 0$ and the most general quadric U which can represent the curve.

It is convenient, for some purposes, to replace the covariants of the system $\sum_2$ by the corresponding apolar forms : *

$$
(5)\qquad F,\ G_1 = G + \frac{I}{2}U',\quad H_1 = H - \frac{J}{4}U',\ Y.
$$

§28. *Binary system of equi-forms.*—The quaternary system $\sum_2$ of the preceding article will now be transformed so as to apply to the cyclic represented by the double binary form

$$f = a_\lambda^2 a_\mu^2,$$

the method being that developed in §21.

For the invariants, the formulæ (5) §20 give †

$$
4I = (pp'AB)^2 = 2\begin{vmatrix} (aa)(\alpha\alpha) & (ab)(\alpha\beta) \\ (ab)(\alpha\beta) & (bb)(\beta\beta)\end{vmatrix} = -2(ab)^2(\alpha\beta)^2;
$$

$$
6J = (pABC)^2 = -\begin{vmatrix} 0 & (ab)(\alpha\beta) & (bc)(\alpha\gamma) \\ (ba)(\beta\alpha) & 0 & (bc)(\beta\gamma) \\ (ca)(\gamma\alpha) & (cb)(\gamma\beta) & 0 \end{vmatrix}
$$

$$
= -2(ab)(bc)(ca)(\alpha\beta)(\beta\gamma)(\gamma\alpha);
$$

$$
\begin{aligned}
24K = (ABCD)^2 &= \{(ac)(bd)(\alpha\beta)(\gamma\delta) - (ab)(cd)(\alpha\gamma)(\beta\delta)\}\\
&= 3(ab)^2(cd)^2(\alpha\beta)^2(\gamma\delta)^2 - 6(ab)(bc)(cd)(da)(\alpha\beta)(\beta\gamma)(\gamma\delta)(\delta\alpha)\\
&= 2(ab)^2(cd)^2(\alpha\gamma)^2(\beta\delta)^2 - 6(ab)(bd)(ac)(cd)(\alpha\beta)(\beta\delta)(\alpha\gamma)(\gamma\delta).
\end{aligned}
$$

---

* That Y is apolar to U', is a special case of the easily proved theorem : the necessary and sufficient condition in order that the product of a number of linear forms shall be apolar to a quadric, is that each linear form shall be conjugate to every other with respect to the quadric. It will appear later (§30) that Y satisfies this condition.

† The D which appears in §20 is here taken as unity for convenience.

Denoting by f, g, h the covariants corresponding to F, G, H, we have, by (5), (6) § 20,

$$f = a_\lambda^2 \alpha_\mu^2, \quad g = (ab)(\alpha\beta)a_\lambda b_\lambda \alpha_\mu \beta_\mu,$$
$$h = \{(ac)(\alpha\beta)b_\lambda\gamma_\mu - (ab)(\alpha\gamma)c_\lambda\beta_\mu\}^2;$$

while Y, the Jacobian of $U'FGH$, becomes (§ 22)

$$Y = \begin{vmatrix} f_{11} & f_{12} & f_{21} & f_{22} \\ g_{11} & g_{12} & g_{21} & g_{22} \\ h_{11} & h_{12} & h_{21} & h_{22} \\ \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \end{vmatrix}.$$

The contravariants are

$\sigma' = (rr')(\rho\rho')$,

$\phi = -(ar)(ar')(\alpha\rho)(\alpha\rho')$,

$\psi = -(ab)(\alpha\beta)(ar)(br')(\alpha\rho)(\beta\rho')$,

$\chi = \{(ac)(\alpha\beta)(br)(\gamma\rho) - (ab)(\alpha\gamma)(cr)(\beta\rho)\}\{(ac)(\alpha\beta)(br')(\gamma\rho') - (ab)(\alpha\gamma)(cr')(\beta\rho')\}$,

$Z =$ Jacobian $(\sigma'\phi\psi\chi)$.

Thus the *equi-binary system of the cyclic*, or the complete system of invariants, contravariants and equi-ordinal covariants of the quadri-quadric

$$f = a_\lambda^2 \alpha_\mu^2,$$

consists of the forms:

$$(\Sigma_3) \qquad \begin{array}{l} I,\ J,\ K, \\ f,\ g,\ h,\ Y, \\ \phi,\ \psi,\ \chi,\ Z,\ \sigma'. \end{array}$$

Of these σ', since it does not depend upon f, may be called the identical contravariant.

§ 29. *Non-symbolic values, and normal form of* $\sum_3$.—In expanded form f is

$$f = (a_{00}\mu_1^2 + 2a_{01}\mu_1\mu_2 + a_{02}\mu_2^2)\lambda_1^2 + 2(a_{10}\mu_1^2 + 2a_{11}\mu_1\mu_2 + a_{12}\mu_2^2)\lambda_1\lambda_2 + (a_{20}\mu_1^2 + 2a_{21}\mu_1\mu_2 + a_{22}\mu_2^2)\lambda_2^2. \tag{1}$$

If we take for the fundamental quadric,

$$U' = 2x_1x_4 - 2x_2x_3, \tag{2}$$

the transformation formulæ may be written

$$x_1 = \lambda_1\mu_1, \quad x_2 = \lambda_1\mu_2, \quad x_3 = \lambda_2\mu_1, \quad x_4 = \lambda_2\mu_2, \tag{3}$$

and the quadric corresponding to f is (§ 18)

$$F = \sum A_{ik} x_i x_k \,,$$

where

$$(4)\qquad \begin{aligned} &A_{11} = a_{00}, \quad A_{12} = a_{01}, \quad A_{13} = a_{10}, \quad A_{14} = a_{11}, \quad A_{22} = a_{02}, \\ &A_{23} = a_{11}, \quad A_{24} = a_{12}, \quad A_{33} = a_{20}, \quad A_{34} = a_{21}, \quad A_{44} = a_{22}. \end{aligned}$$

The concomitants $\sum_3$ of f may then be obtained directly from the concomitants of U', F. The invariants are obtained by forming the discriminant of $U' + \rho F$:

$$(5)\quad D(\rho) = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} + \rho \\ A_{21} & A_{22} & A_{23} - \rho & A_{24} \\ A_{31} & A_{32} - \rho & A_{33} & A_{34} \\ A_{41} + \rho & A_{42} & A_{43} & A_{44} \end{vmatrix} = \begin{vmatrix} a_{00} & a_{01} & a_{10} & a_{11} + \rho \\ a_{01} & a_{02} & a_{11} - \rho & a_{12} \\ a_{10} & a_{11} - \rho & a_{20} & a_{21} \\ a_{11} + \rho & a_{12} & a_{21} & a_{22} \end{vmatrix}$$
$$= \rho^4 + I\rho^2 + J\rho + K.$$

The expansion gives

$$(6)\qquad \begin{aligned} I &= 2(a_{10}a_{12} + a_{01}a_{21} - a_{11}^2) - a_{00}a_{22} - a_{02}a_{20}, \\ J &= 2\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}, \quad K = \begin{vmatrix} a_{00} & a_{01} & a_{10} & a_{11} \\ a_{01} & a_{02} & a_{11} & a_{12} \\ a_{10} & a_{11} & a_{20} & a_{21} \\ a_{11} & a_{12} & a_{21} & a_{22} \end{vmatrix}. \end{aligned}$$

Similarly, the covariants may be obtained from

$$(7)\qquad \begin{vmatrix} 0 & \lambda_2\mu_2 & -\lambda_2\mu_1 & -\lambda_1\mu_2 & \lambda_1\mu_1 \\ \lambda_2\mu_2 & a_{00} & a_{01} & a_{10} & a_{11} + \rho \\ -\lambda_2\mu_1 & a_{01} & a_{02} & a_{11} - \rho & a_{12} \\ -\lambda_1\mu_2 & a_{10} & a_{11} - \rho & a_{20} & a_{21} \\ \lambda_1\mu_1 & a_{11} + \rho & a_{12} & a_{21} & a_{22} \end{vmatrix} = \rho^2 f - \rho g - h,$$

and the contravariants from

$$(8)\qquad \begin{vmatrix} 0 & r_{11} & r_{12} & r_{21} & r_{22} \\ r_{11} & a_{00} & a_{01} & a_{10} & a_{11} + \rho \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \rho^3\sigma' + \rho^2\phi + \rho\psi + \chi.$$

It will suffice, for later applications, to write down the explicit values of the covariants and contravariants only for a certain normal form which will now be considered. The normal quaternary form of the cyclic is

$$\text{[A]} \quad \begin{cases} U' = \sum y_i^2, \\ F = \sum \beta_i y_i^2, \end{cases} \qquad (\Sigma\beta_i = 0).$$

This, by the substitution:

$$(9) \quad \begin{aligned} \sqrt{2}y_1 &= x_1 + x_4, & \sqrt{2}x_1 &= y_1 + cy_4, \\ i\sqrt{2}y_2 &= x_2 + x_3, & \sqrt{2}x_2 &= y_3 + cy_2, \\ \sqrt{2}y_3 &= x_2 - x_3, & \sqrt{2}x_3 &= y_3 - cy_2, \\ i\sqrt{2}y_4 &= x_1 - x_4, & \sqrt{2}x_4 &= y_1 - cy_4, \end{aligned}$$

becomes

$$\text{[B]} \quad \begin{cases} U' = 2x_1x_4 - 2x_2x_3, \\ F = k(x_1^2 + x_4^2) + l(x_2^2 + x_3^2) + 2m(x_1x_4 + x_2x_3), \end{cases}$$

where

$$(10) \quad \begin{aligned} 2k &= \beta_1 - \beta_4, & \beta_1 &= m + k, \\ 2l &= \beta_3 - \beta_2, & \beta_2 &= -m - l, \\ 2m &= \beta_1 + \beta_4, & \beta_3 &= -m + l, \\ &= -\beta_2 - \beta_3, & \beta_4 &= m - k. \end{aligned}$$

Finally, by (3),

$$\text{[C]} \qquad f = k\lambda_1^2\mu_1^2 + l_1\lambda_1^2\mu_2^2 + 4m\lambda_1\mu_1\lambda_2\mu_2 + l\mu_1^2\lambda_2^2 + k\lambda_2^2\mu_2^2.$$

The general cyclic can be reduced to any of the normal forms [A], [B], [C]; each contains two essential constants as it should, since the cyclic has $8 - 6 = 2$ absolute invariants.

For the binary normal form [C], the system $\sum_3$ is

$$(11) \quad \begin{aligned} &I = -(k^2 + l^2 + 2m^2), \quad J = 2m(k^2 - l^2), \quad K = (m^2 - k^2)(m^2 - l^2); \\ &f = k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) + 4m\lambda_1\lambda_2\mu_1\mu_2, \\ &g = 2km(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) - 2lm(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) + 2(k^2 - l^2)\lambda_1\lambda_2\mu_1\mu_2, \\ &h = k(l^2 - m^2)(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(k^2 - m^2)(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) \\ &\qquad\qquad - 2m(k^2 + l^2 - 2m^2)\lambda_1\lambda_2\mu_1\mu_2, \\ &Y = -\tfrac{1}{4}\sqrt{\delta}(\lambda_1^2\mu_1^2 - \lambda_2^2\mu_2^2)(\lambda_1^2\mu_2^2 - \lambda_2^2\mu_1^2); \end{aligned}$$

$$
(11)\quad
\begin{aligned}
\sigma' &= 2r_{11}r_{22} - 2r_{12}r_{21}\,,\\
\phi &= -\,k(r_{11}^2 + r_{22}^2) - l(r_{12}^2 + r_{21}^2) - 2m(r_{11}r_{22} + r_{12}r_{21})\,,\\
\psi &= 2km(r_{11}^2 + r_{22}^2) - 2lm(r_{12}^2 + r_{21}^2) - 2(l^2 + m^2)r_{11}r_{22} + 2(k^2 + m^2)r_{12}r_{21}\,,\\
\chi &= k(l^2 - m^2)(r_{11}^2 + r_{22}^2) + l(k^2 - m^2)(r_{12}^2 + r_{21})^2\\
&\qquad - 2m(l^2 - m^2)r_{11}r_{22} - 2m(k^2 - m^2)r_{12}r_{21}\,,\\
Z &= -\tfrac{1}{4}\sqrt{\delta}(r_{11}^2 - r_{22}^2)(r_{12}^2 - r_{21}^2)\,,
\end{aligned}
$$

where

$$\sqrt{\delta} = 4kl\{(k^2 + l^2)^2 - 16m^4\}\,.$$

§ 30. *The polars and the covariants.*—If we apply the definition of § 20 to the cyclic

$$f = a_\lambda^2 a_\mu^2\,,$$

we see that every point $P = (\xi,\ \eta)$ has five polars Q_{10}, Q_{01}, Q_{11}, Q_{12}, Q_{21}, whose expressions and geometric definitions follow.

The polar minimal lines,

$$L' = a_\xi a_\eta^2 a_\lambda = 0\,,\quad M' = a_\xi^2 a_\eta a_\mu = 0$$

are the harmonics of ξ (or η) with respect to the λ-lines (or μ-lines) which correspond to η (or ξ) in f. The polar circle

$$C' = a_\xi a_\eta a_\lambda a_\mu = 0\,,$$

is generated by the following correspondence: to each μ corresponds the harmonic of ξ with respect to that pair of λ's, to each of which corresponds in f a pair of μ's harmonic to η and μ; and to each λ, etc. The polar circle, then, cuts each of the minimal lines through P in the harmonic of P with respect to the intersections of the minimal line and the cyclic; and the conjugate point P', of P with respect to f, which is defined as the intersection of L' and M', is the inverse of P with respect to C'. The polars,

$$Q_{21} = a_\eta a_\lambda^2 a_\mu\,,\quad Q_{12} = a_\xi a_\lambda a_\mu^2\,,$$

are the loci of points whose polar μ-lines and λ-lines, respectively, pass through P; their intersections give the five points whose conjugates coincide with P.

It is convenient in the discussion of the relation of the polars and covariants to consider the cyclic in its quaternary normal form [A] § 29. If the coördinates of P are x_i,

	the coördinates of C'	are	$\beta_i x_i$,
(1)	" " " C''	"	$\beta_k\beta_l\beta_m x_i$,
	and " " " P'	"	$(2\beta_i F - G)x_i$,

where C'' is the "anti-polar" circle of P, i. e., the circle orthogonal to the polar circles of all points on the minimal lines through P. The polar and anti-polar circles of any point are orthogonal; for

$$I(C'C'') = \sum \beta_1\beta_2\beta_3\beta_4 x_i^2 = 0 .$$

The invariants of P, C', C'', P'' are

$$(2)\qquad \begin{aligned} &I(PC') = F, \quad I(PC'') = H, \quad I(C'C') = G, \quad I(C'C'') = 0, \\ &I(C'P') = FG, \quad I(C''C'') = JH - KG, \quad I(P'C'') = -\,GH, \end{aligned}$$

as may be proved by (3) § 27. From these we obtain the following interpretations.

The curve $F = 0$ is the given cyclic; its points are characterized by the fact that the polar circle of each passes through it.

The curve $G = 0$ is the locus of a point whose polar circle with respect to F is degenerate; it is also the locus of the vertices of degenerate apolar circles.

Finally, the locus of a point whose anti-polar circle passes through it, is $H = 0$.

If $\overline{P}$, C', C'' have a common point, then

$$\begin{vmatrix} 0 & F & H \\ F & G & 0 \\ H & 0 & JH - KG \end{vmatrix} = F^2(KG - JH) - G^2H = 0 ;$$

and if the polar and anti-polar circles are tangent,

$$T(KG - JH) = 0 .$$

The conjugate of the conjugate of P has the coördinates:

$$[2\beta_i\{ + F^2(IF + H) + 3FG^2\} + 4F^2(JF + IG) + 4FG(IF + G) + G^3]x_i ,$$

so that the locus of a point whose second conjugate lies on the same minimal line is

$$2F^2\{4F(IF + H) + 3G^2\} = 0 .$$

Consider now the relations of the polar circles of a point with respect to F, G, H; these circles (§ 19) are to be found from the apolar forms of the covariants as given in (5) § 27. The coördinates are as follows:

$$(4)\qquad \begin{aligned} &P: \quad x_i , \quad C_G: \left(\beta_i^2 + \frac{I}{2}\right)x_i , \\ &C_F: \beta_i x_i , \quad C_H: \left(\beta_k\beta_l\beta_m - \frac{J}{4}\right)x_i . \end{aligned}$$

Their invariants are

$$
\begin{aligned}
&I(PC_F) = F, \quad I(C_F C_G) = -\frac{I}{2}F - H, \quad I(C_F C_H) = -\frac{J}{4}F,\\
(5)\qquad &I(C_G C_G) = JF, \quad I(C_G C_H) = KF - \frac{J}{4}G + \frac{I}{2}H,\\
&I(C_H C_H) = \frac{J}{2}H - KG.
\end{aligned}
$$

From these we obtain the (1, 1) transvectants of the corresponding double binary forms f, g, h [(1) §14]:

$$
(6)\qquad
\begin{aligned}
&(ff)_{11} = I(C_f C_f) = g, \quad (fg)_{11} = -\frac{I}{2}f - h, \quad (fh)_{11} = -\frac{J}{4}f,\\
&(gg)_{11} = Jf, \quad (gh)_{11} = Kf - \frac{J}{4}g - \frac{I}{2}h, \quad (hh)_{11} = \frac{J}{2}h - Kg;
\end{aligned}
$$

and the substitution of these values in (2) §14 gives the reduction of Y^2 as follows:

$$
Y^2 = \begin{vmatrix}
g & -\frac{I}{2}f - h & -\frac{J}{4}f & f\\
-\frac{I}{2}f - h & Jf & Kf - \frac{J}{4}g - \frac{I}{2}h & g\\
-\frac{J}{4}f & Kf - \frac{J}{4}g - \frac{I}{2}h & \frac{J}{2}h - Kg & h\\
f & g & h & 0
\end{vmatrix}.
$$

From §14 and the normal form (3) §27, we have:

The curve $Y = 0$ is the locus of a point which lies on the orthogonal circle of its polar circles with respect to f, g and h; this locus consists of four mutually orthogonal circles, the "director" circles of the cyclic.*

§ 31. *The discriminants.*—As defined in § 15, the discriminants are

$$
\begin{aligned}
D_1(\lambda) &= \tfrac{1}{2}(ff)_{20} = \tfrac{1}{2}(\alpha\beta)^2 a_\lambda^2 b_\lambda^2,\\
D_2(\mu) &= \tfrac{1}{2}(ff)_{02} = \tfrac{1}{2}(ab)^2 \alpha_\mu^2 \beta_\mu^2;
\end{aligned}
$$

and for the normal form [C] § 29 they are

$$
(1)\qquad
\begin{aligned}
D_1(\lambda) &= -kl\lambda^4 + (4m^2 - k^2 - l^2)\lambda^2 - kl,\\
D_2(\mu) &= -kl\mu^4 + (4m^2 - k^2 - l^2)\mu^2 - kl.
\end{aligned}
$$

* The covariant curves cut f in the following points: g, in the 8 minimal points (§13); h, in the 8 points where the polar circle is also the osculating circle (§23); Y, in the 16 points where the osculating circle has 4-point contact (CLEBSCH, Crelle, vol. 63, p. 9, gives the space theorem corresponding to the last result).

Calculating the invariants i, j of these biquadratics and comparing with the values of I, J, K in (11) § 29, we find

$$(2) \qquad 6i_1 = 6i_2 = I^2 + 12K, \quad 36j_1 = 36j_2 = -I^3 + 36IK - 6J^2;$$

these are also the invariants of

$$D(\rho) = \rho^4 + I\rho^2 + J\rho + K,$$

where $D(\rho)$ is equal to the characteristic determinant of the two quadrics U', F. This gives FROBENIUS' generalization of CAYLEY'S theorem:*

The discriminants D_1, D_2 and the characteristic biquadratic D, have equal invariants.

Equated to 0, $D_1(\lambda)$ and $D_2(\mu)$ represent the four λ-tangents and the four μ-tangents of the cyclic; Cayley's theorem then states that these two sets of lines have equal anharmonic ratios.

The minimal points of f, i. e., the points of contact of the minimal tangents $D_1(\lambda) = 0$, $D_2(\mu) = 0$, are cut out (§ 13) by the $(1, 1)$ transvectant

$$(ff)_{11} = g = 0;$$

but from (6) § 30,

$$(gg)_{11} = Jf;$$

therefore the minimal points of f are also the minimal points of g. The discriminants of g:

$$E_1(\lambda) = \tfrac{1}{2}(gg)_{20}, \quad E_2(\mu) = \tfrac{1}{2}(gg)_{02},$$

have then the interpretation: $E_1(\lambda) = 0$ represents the λ-lines through the points of contact of the μ-tangents $D_2(\mu) = 0$; and $E_2(\mu) = 0$, the μ-lines through the points of contact of the λ-tangents $D_1(\lambda) = 0$.

The normal form $[C]$ of E_1 is

$$(3) \qquad E_1(\lambda) = 4klm^2\lambda^4 + \{(k^2 - l^2)^2 - 4m^2(k^2 + l^2)\}\lambda^2 + 4klm^2;$$

comparing this with D_1 and its Hessian H_{D_1}, we obtain the following for E_1 (and similarly for E_2):

$$(4) \qquad E_1 = \frac{2I}{3} D_1 - 2H_{D_1}, \quad E_2 = \frac{2I}{3} D_2 - 2H_{D_2};$$

it follows that the quadruples of λ-lines represented by D_1 and E_1 are not independent, but connected by the fact that the sextic covariants of both represent the same six lines.†

---

* CAYLEY'S theorem states the equality of the invariants of D_1, D_2; CAYLEY, Quarterly Journal, vol. 11, p. 83-91; other proofs have been given by CAPELLI, ZEUTHEN, LEPAIGE; FROBENIUS' generalization, Crelle's Journal, vol. 106, p. 129, 1890.

† CLEBSCH, *Binäre Formen*, § 51.

The application of the formulæ in CLEBSCH's *Binäre Formen* § 41, gives the concomitants of E_1 as follows:

$$(5)\quad \begin{aligned} 9i' &= 4I^2i - 24Ij + 6i^2\,, \\ 27j' &= 8I^3j - 12I^2i^2 + 36Iij - 72j^2 + 6i^3\,, \\ H_{E_1} &= \tfrac{1}{3}(4j - 2iI + \tfrac{4}{9}I^3)D_1 - \tfrac{1}{6}(\tfrac{4}{3}I^2 - 2i)E_1\,, \\ S_{E_1} &= -J^2S_{D_1}\,, \end{aligned}$$

where S represents the sextic covariant of the corresponding biquadratic.

The λ-tangents $D_1(\lambda) = 0$ and the μ-tangents $D_2(\mu) = 0$ intersect in the sixteen foci of the cyclics; and from Cayley's theorem it is easy to show that these foci lie by fours on the four director circles. The director circles of f are given by $Y = 0$ (§ 30), where Y is the Jacobian of U', F, G, H; the totality of cyclics, therefore, which have the same director circles as f, is the net

$$(6)\qquad t_1f + t_2g + t_3h = 0\,.$$

Included in this net is the system of cyclics confocal with f,

$$(7)\qquad f - \rho g - \rho^2h\,;$$

through each point of the plane there pass in general two orthogonal cyclics of the confocal system, the exceptional points (for which the two fall together) lying on the eight minimal tangents:*

$$g^2 + 4fh = 0 \quad \text{or} \quad D_1(\lambda)D_2(\mu) = 0\,.$$

§ 32. *Contravariants.*—The definition of polar and anti-polar circles given in § 30, may be extended so as to apply to a circle instead of to a point: the polar circle C_1 of a circle C, with respect to a cyclic f, is the locus of points whose polar circles are orthogonal to C; the anti-polar circle C_2 of C is the circle orthogonal to the polar circles of all the points of C. For the normal form the coordinates of these circles are

$$(1)\qquad C:\ u_i\,, \quad C_1:\ \beta_iu_i\,, \quad C_2:\ \beta_k\beta_l\beta_mu_i\,,$$

from which it can be shown that the polar circle of the anti-polar circle (as also the anti-polar circle of the polar circle) coincides with the original circle. The invariants of (1) are

$$(2)\quad \begin{aligned} &I(CC) = \sum u^2 = \sigma', \quad I(CC_1) = \sum \beta_iu_i^2 = -\phi\,, \quad I(CC_2) = \sum \beta_k\beta_l\beta_mu_i^2 = \chi\,, \\ &I(C_1C_1) = \sum \beta_i^2u_i^2 = \psi - I\sigma', \quad I(C_1C_2) = K\sigma', \quad I(C_2C_2) = J\chi - K\psi\,. \end{aligned}$$

*See (6) § 32.

Therefore

$\sigma' = 0$ is the quadratic complex of degenerate circles ;

$\phi = 0$ is the quadratic complex of circles orthogonal to their polar circles;

$\psi - I\sigma' = 0$ is the quadratic complex of circles whose polar circles are degenerate;

$\chi = 0$ is the quadratic complex of circles orthogonal to their anti-polar circles.

Finally Z, being the Jacobian of σ', ϕ, ψ, χ, represents the circles belonging to the linear system determined by their polars with respect to f, g, h; it consists of the circles orthogonal to the directors circles.

The polar circles of all the points of the plane form the complex $J\chi - K\psi = 0$, while the anti-polar circles form $\psi - I\sigma' = 0$.

Applying the first principle of § 16, the intersections of $C = r_\lambda \rho_\mu$ and $f = a_\lambda^2 a_\mu^2$ are given by

$$L = (fC^2)_{02} = (a\rho)(a\rho')a_\lambda^2 r_\lambda r'_\lambda,$$

$$M = (fC^2)_{20} = (ar)(ar')a_\mu^2 \rho_\mu \rho'_\mu .$$

Therefore, *the complex of circles, which cut f in four points whose anharmonic ratio on the circle is a*, is given by *

$$\frac{i_L^3}{j_L^2} = \frac{i_M^3}{j_M^2} = 24 \frac{(1 - a + a^2)^3}{(1 + a)^2(2 - a)^2(1 - 2a)^2}. \tag{3}$$

In general the complex is therefore of the twelfth class ; but for the equianharmonic and harmonic cases, it reduces to

$$i_L = 0, \quad j_L = 0,$$

which are of the fourth and of the sixth classes respectively ; and for the coincident case the complex decomposes into [(4) § 16]

$$D_L = i_L^3 - 6j_L^2 = \Phi\sigma'^2 .$$

The *tangential complex* Φ, expressed in terms of the fundamental contravariants, is †

$$\Phi = 4(3\sigma'\psi - \phi^2)(3\phi\chi - \psi^2) - (9\sigma'\chi - \phi\psi)^2. \tag{4}$$

The osculating circles belong to both the complexes $i_L = 0$, $j_L = 0$.

* In terms of the fundamental contravariants, this comeplx is $(4\Phi^2 - 6I\sigma'^2)^3 + A\sigma'^2\Phi$, where A is a function of the anharmonic ratio a, and Φ is defined by (4).

† SALMON-FIEDLER, *Raumgeometrie*, 3d edition, vol. II, p. 345.

Each contravariant (§ 16) yields a corresponding covariant—the locus of the vertices of the degenerate circles contained in the complex ; thus, from ϕ, ψ, χ, Z we obtain F, G, H, Y. Substituting these in (3), the corresponding covariant is found to be

$$3F^2(G^2 + 4FH);$$

but it is evident that a degenerate circle, whose centre is not on F, can be tangent to F only when one of its minimal lines is tangent to F; so that

$$G^2 + 4FH = 0 \tag{4}$$

represents the minimal tangents of F. These tangents are also given by

$$D_1(\lambda)D_2(\mu) = 0\,; \tag{5}$$

in fact from the normal forms it can be shown that

$$g^2 + 4fh = -4D_1D_2. \tag{6}$$

§ 33. *Interpretations for the invariants.**—The application of formula (4) § 24 gives the following values of the (2, 2) transvectants:

$$\begin{aligned} &(ff)_{22} = -2I, \quad (fg)_{22} = 3J, \quad (fh)_{22} = 4K, \\ &(gg)_{22} = I^2 - 4K, \quad (gh)_{22} = \frac{IJ}{2}, \quad (hh)_{22} = \tfrac{3}{4}J^2 - 2IK. \end{aligned} \tag{1}$$

From the conclusion of § 24 we have then:

If $I = 0$, the complex ϕ contains an infinite number of quadruples mutually conjugate with respect to f, and conversely if ϕ contains one such set (and therefore an infinity), $I = 0$; similarly $J = 0$, $K = 0$ are the necessary and sufficient conditions for ϕ containing quadruples mutually conjugate with respect to g and h respectively.

Other interpretations are obtained from the consideration of the covariant quadrics in space. If $I = 0$, the complex ψ contains quadruples of mutually orthogonal circles. If $J = 0$, there exist self-polar point quadruples, i. e., sets of four points possessing the property that the polar circle of each passes through the other three ; and χ contains mutually orthogonal quadruples of circles. If $K = 0$, the polar circles of all the points of the cyclic are orthogonal to a fixed circle, which circle is one of the director circles ; on this director circle there exist triples of points whose polar circles are mutually conjugate ; and χ degenerates into the linear complex (counted twice) of circles orthogonal to the special director circle.

* See also § 37.

§ 34. *Complete binary system.*—In § 28 a system of forms $\sum_3$ of the quadri-quadric

$$f = a_\lambda^2 a_\mu^2 ,$$

was obtained, in terms of which any invariant, concomitant or equi-covariant can be rationally and integrally expressed; if we wish to consider also co-variants whose partial orders are unequal, we must add to this system the forms of § 28 and the $(1, 0)$ and $(0, 1)$ transvectants,

$$(1) \qquad \begin{aligned} \rho_1 &= (gh)_{01}, \quad \rho_2 = (hf)_{01}, \quad \rho_3 = (fg)_{01}, \\ \pi_1 &= (gh)_{10}, \quad \pi_2 = (hf)_{10}, \quad \pi_3 = (fg)_{10}, \end{aligned}$$

where (§ 13) $\pi_1 = 0$ ($\rho_1 = 0$) represents the loci of points whose polar λ-lines (μ-lines) with respect to g and h coincide; and $\rho_1 = 0$, $\pi_1 = 0$ intersect in the 18 points which have the same conjugates with respect to either f or g, etc. The completeness of the system thus obtained is proved by a combination of the results already obtained, and a comparison with PEANO'S system of in- and co-variants; for this comparison it is convenient to express the forms as transvec-tants.

From (1) §33 the invariants may be written

$$I = -\tfrac{1}{2}(ff)_{22}, \quad J = \tfrac{1}{3}(fg)_{22}, \quad K = \tfrac{1}{4}(fh)_{22} .$$

The covariants are*

$$\begin{aligned} &f, \quad g = (ff)_{11}, \quad h = -(fg)_{11} + \tfrac{1}{4}(ff)_{22} \cdot f, && \text{from (6) § 30;} \\ &\pi_1 = (gh)_{10}, \quad \pi_2 = (hf)_{10}, \quad \pi_3 = (fg)_{10}, && \\ &\rho_1 = (gh)_{01}, \quad \rho_2 = (hf)_{01}, \quad \rho_3 = (fg)_{01}, && \text{from (1);} \\ &D_1 = \tfrac{1}{2}(ff)_{02}, \quad E_1 = \tfrac{1}{2}(gg)_{02}, \quad S_1 = (D_1E_1)_{10}, && \\ &D_2 = \tfrac{1}{2}(ff)_{20}, \quad E_2 = \tfrac{1}{2}(gg)_{20}, \quad S_2 = (D_2E_2)_{01}, && \text{from § 31.} \end{aligned}$$

The contravariants (2) § 32 are

$$\begin{aligned} &\sigma' = (CC)_{11}, \quad \phi = -(CC_1)_{11}, \quad \psi = (C_1C_1)_{11} + I(CC)_{11}, \\ &\chi = (CC_2)_{11}, \quad Z = \text{Jacobian } (\sigma'\phi\psi\chi) ; \end{aligned}$$

where

$$C_1 = (fC)_{11}, \quad C_2 = (hC)_{11} + \frac{J}{4}C, \quad C = r_\lambda \rho_\mu .$$

The invariants and contravariants are complete by § 32, and the completeness of the covariants is shown by their equivalence to PEANO'S covariants.† The

*Y is omitted since it is expressible integrally through the other covariants.

† PEANO, Giornale di Matematiche, vol. 20, p. 97, 1882; contravariants are not considered. GORDAN, Mathematische Annalen, vol. 23, p. 388, 1889, gives a complete system of 38 invariants and covariants; but this is reducible to PEANO'S 18 forms and therefore also to the forms in the text.

complete binary system consists then of the twenty-three forms:

$$(\Sigma_4)\qquad \begin{array}{cccccc} I, & J, & K; & f, & g, & h; \\ \pi_1, & \pi_2, & \pi_3, & \rho_1, & \rho_2, & \rho_3; \\ D_1, & E_1, & S_1, & D_2, & E_2, & S_2; \\ \sigma', & \phi, & \psi, & \chi, & Z. & \end{array}$$

§35. *Equivalence and classification of the cyclics.*—The character of the cyclic:

$$(1)\qquad F = A_x^2 = 0, \quad U' = p_x^2 = 0,$$

or

$$(2)\qquad f = a_\lambda^2 a_\mu^2 = 0,$$

depends essentially upon the determinant:

$$(3)\qquad D(\rho) = |A_{ik} + \rho p_{ik}|,$$

or, by (5) § 29,

$$(4)\qquad D(\rho) = \begin{vmatrix} a_{00} & a_{01} & a_{10} & a_{11}+\rho \\ a_{01} & a_{02} & a_{11}-\rho & a_{12} \\ a_{10} & a_{11}-\rho & a_{20} & a_{21} \\ a_{11}+\rho & a_{12} & a_{21} & a_{22} \end{vmatrix}.$$

Applying Weierstrass's theory of quadratic forms * we have:

Two cyclics are equivalent † when and only when their characteristic determinants have proportional elementary divisors.

This condition may be decomposed into a qualitative and quantitative: 1° similarity of the characteristic determinants, i. e., the elementary divisors of both characteristic determinants occur to the same degrees; 2° equality of absolute invariants, i. e., the roots of the characteristic biquadratic

$$(5)\qquad D(\rho) = \rho^4 + I\rho^2 + J\rho + K = (\rho - \beta_1)(\rho - \beta_2)(\rho - \beta_3)(\rho - \beta_4)$$

of both characteristic determinants are proportional.

If we call two cyclics of the same "species" when they satisfy the condition 1°, we have in all thirteen species; these, as defined by the symbol ‡ of the char-

* Weierstrass, *Berliner Monatsberichte*, 1868.

† Since every cyclic can be transformed into itself by both proper and improper circular transformations (in general four of each kind), equivalence in the group G and equivalence in the group G' are (for cyclics) identical. Therefore the theorem above applies to both groups.

‡ The notation is that introduced by Weiter in his classification of the quadratic line-complexes, Mathematische Annalen, vol. 7; Ib or [(11)11], for example, signifies that the characteristic determinant has two simple and one double factor, the latter also appearing as a simple factor in all the first minors. The case [(1111)] is omitted as trivial since it can occur only when $f \equiv 0$.

acteristic determinant, are given in the following table:

(6)

	a	b	c	d
I	[1111]	[(11)11]	[(11)(11)]	[(111)1]
II	[211]	[(21)1]	[2(11)]	[(211)]
III	[31]	[(31)]		
IV	[22]	[(22)]		
V	[4]			

In order to study the geometric peculiarities of the different species, it is sufficient to consider them in their normal forms. The quaternary normal forms for a pair of quadrics can be obtained by Weierstrass' methods;* reducing the results to apolar form we obtain the quaternary normal forms of the cyclic as follows:

$$\text{I} \quad \begin{cases} U' = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ F = \beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_4^2 \end{cases} \qquad (\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0);$$

$$\text{II} \quad \begin{cases} U' = x_1^2 + x_4^2 - 2x_2x_3, \\ F = \beta_1 x_1^2 + \beta_2 x_4^2 - 2\beta_3 x_2 x_3 + x_2^2 \end{cases} \qquad (2\beta_3 + \beta_1 + \beta_2 = 0);$$

$$\text{III} \quad \begin{cases} U' = x_1^2 + x_4^2 - 2x_2x_3, \\ F = \beta_1 x_1^2 + \beta_2(x_4^2 - 2x_2x_3) + 2(\sqrt{2}x_2x_3) \end{cases} \qquad (3\beta_2 + \beta_1 = 0);$$

$$\text{IV} \quad \begin{cases} U' = 2x_1x_4 - 2x_2x_3, \\ F = 2\beta_1 x_1 x_4 - 2\beta_2 x_2 x_3 + x_3^2 - x_4^2 \end{cases} \qquad (2\beta_1 + 2\beta_2 = 0);$$

$$\text{V} \quad \begin{cases} U' = 2x_1x_4 - 2x_2x_3, \\ F = 2\beta_1(x_1x_4 - x_2x_3) + x_2^2 + 2x_1x_3 \end{cases} \qquad (4\beta_1 = 0).$$

These are the normal forms for Ia, IIa, IIIa, IVa, Va, the remaining species being obtained from these by equating the proper β's.

Applying now the method of § 29, we obtain the following normal forms of the thirteen species of quadri-quadrics:

Ia, $\quad k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(\lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2) + 4m\lambda_1\lambda_2\mu_1\mu_2$;

Ib, $\quad l(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) - 4m\lambda_1\lambda_2\mu_1\mu_2$;

Ic, $\quad 4\lambda_1\lambda_2\mu_1\mu_2$;

Id, $\quad \lambda_1^2\mu_2^2 + \lambda_2^2\mu_1^2 - 2\lambda_1\lambda_2\mu_1\mu_2$;

* Without the use of Weierstrass' general formulæ, corresponding normal forms are obtained in CLEBSCH-LINDEMANN, *Geometrie*, vol. II, p. 202 ff.; for conics compare GUNDELFINGER, *Kegelschnitte*, §16.

IIa, $\quad k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + \lambda_1^2\mu_2^2 + 4m\lambda_1\lambda_2\mu_1\mu_2$;

IIb, $\quad \lambda_1^2\mu_2^2 - 2m(\lambda_1\mu_1 - \lambda_2\mu_2)^2$;

IIc, $\quad \lambda_1^2\mu_2^2 + 4m\lambda_1\lambda_2\mu_1\mu_2$;

IId, $\quad \lambda_1^2\mu_2^2$;

IIIa, $\quad k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2 + 2\lambda_1\lambda_2\mu_1\mu_2) + 2\lambda_1\mu_1(\lambda_1\mu_1 - \lambda_2\mu_2)$;

IIIb, $\quad 2\lambda_1\mu_2(\lambda_1\mu_1 - \lambda_2\mu_2)$;

IVa, $\quad \lambda_2^2\mu_1^2 + 4m\lambda_1\lambda_2\mu_1\mu_2 - \lambda_2^2\mu_2^2$;

IVb, $\quad \lambda_2^2(\mu_1^2 - \mu_2^2)$;

Va, $\quad \lambda_1^2\mu_2^2 + 2\lambda_1\lambda_2\mu_1^2$.

From these normal forms a large number of results may be derived, some of which are given in the following table:

		$D(\rho)$	$D_1(\lambda)$	$D_2(\mu)$		
I*a*	[1111]	{1111}	{1111}	{1111}	0	general C_{22}
I*b*	[(11)11]	{211}	{22}	{22}	2	$C_{11}C_{11}$
I*c*	[(11)(11)]	{22}	{22}	{22}	4	$C_{10}C_{10}C_{01}C_{01}$
I*d*	[(111)1]	{31}	iden.*	iden.*	5	C_{11}^2
II*a*	[211]	{211}	{211}	{211}	1	nodal C_{22}
II*b*	[(21)1]	{31}	{4}	{4}	3	C_{11} tangent C_{11}
II*c*	[2(11)]	{22}	{22}	{22}	3	$C_{11}C_{10}C_{01}$
II*d*	[(211)]	{4}	iden.*	iden.*	6	$C_{10}^2C_{01}^2$
III*a*	[31]	{31}	{31}	{31}	2	cuspidal C_{22}
III*b*	[(31)]	{4}	{4}	{4}	4	$C_{10}C_{01}$ intersecting on C_{11}
IV*a*	[22]	{22}	{211}	{22}	2	$C_{12}C_{10}$
IV*b*	[(22)]	{4}	{4}	iden.*	5	$C_{10}^2C_{01}C_{01}$
V*a*	[4]	{4}	{31}	{4}	3	C_{12} tangent C_{10}

The first column gives the species; the second, the symbol or "character" of the characteristic determinant; the next three, the characters of the biquadratics D, D_1, D_2; the sixth, the degree of speciality, or the number of conditions necessary in order that a cyclic may belong to the corresponding species; and the last, a description of the form of the curve. For example, consider the species I*b* or [(11)11]: the curve consists of two non-degenerate non-tangent circles; the characteristic biquadratic $D(\rho)$ has a double root; each of the discriminants has two double roots; and the species is doubly special. Some conclusions which can be drawn from this table will now be given.

* This means that the corresponding biquadratic vanishes identically.

The species which represent a pair of circles (proper or degenerate) are Ib, Ic, Id, IIb, IIc, IId, IIIb, IVb; for these (and for no others), both D_1 and D_2 are perfect squares. Therefore

The necessary and sufficient condition that $f = 0$ shall represent a pair of circles, is that both discriminants shall be perfect squares.

In terms of the fundamental concomitants the conditions may be expressed thus:

$$3iE_1 + (6j - 2iI)D_1 \equiv 0,$$

$$3iE_2 + (6j - 2iI)D_2 \equiv 0.$$

In § 31, it was shown that the three biquadratics $D(\rho)$, $D_1(\lambda)$, $D_2(\mu)$, have equal invariants i, j; it does not follow from this that they are equivalent, but it does follow that if their "character" is the same (i. e., if the same equalities hold among their roots), they will be necessarily equivalent. The table then gives the following relations between D, D_1, D_2:*

If the cyclic is irreducible, all three biquadrics are equivalent; in all cases two of the three are equivalent. The necessary and sufficient condition for the equivalence of the discriminants D_1, D_2 is the presence of a linear elementary divisor in the characteristic determinant of f. Translated into geometrical terms, this reads: *except when the cyclic degenerates into either a minimal line and a C_{12} or C_{12}, or into a double minimal line of one system and a pair of minimal lines of the other system—the two pencils of minimal tangents are homographic.*

For the different species, the following relations exist among the invariants:

Ia,	$\delta \neq 0$;
IIa, Ib,	$\delta = 0$;
IIIa, IIb, Id,	$i = 0$, $j = 0$;
IVa, IIc, Ic,	$J = 0$, $4K - I^2 = 0$;
Va, IIIb, IVb, IId,	$I = 0$, $J = 0$, $K = 0$.

Therefore $\delta = i^3 - 6j^2 = 0$ is the condition for a node (this may also be obtained from (2) § 16, since for the cyclic there are no P, Q invariants); and $i = 0$, $j = 0$ are the conditions for a cusp.†

* Cf. FROBENIUS, Crelle, vol. 106, pp. 125–188, 1890. The problem of finding the forms when the discriminants are assigned is there studied; geometrically, this is equivalent to finding the system of cyclics with assigned foci.

† The explicit values of i, j have been calculated by CAYLEY, *Two Invariants of the Quadro-quadric*, 1893, *Works*, vol. XIII. Cf. *Porism of the in- and circumscribed polygon*, 1870, vol. VIII.

CHAPTER VI.

Double Ternary and other Methods.

§ 36. *Double ternary method.*—In § 10 a minimal line of the plane, or a generator of the fundamental quadric, was defined by a parameter λ, or a binary set $\lambda_1 : \lambda_2$; these coördinates will now be replaced by ternary homogeneous coordinates connected by a quadratic identity. One method of introducing these is to establish a $(1, 1)$ correspondence between the pencil of lines considered and the points of a conic: the trilinear coördinates $r_1 r_2 r_3$ of the points may then be taken as the coördinates of the corresponding lines. Again, by the use of line coördinates, the r's may be defined as follows: let the equations, in line coördinates, of three fixed generators of the first system (or more generally of three fixed linear line-complexes, each of which includes all the generators of the second system) be $l' = 0$, $l'' = 0$, $l''' = 0$; the equation of any other generator of the first system is then of the form

$$(1) \qquad r_1 l' + r_2 l'' + r_3 l''' = 0;$$

these parameters r_i may then be called the ternary coördinates of the line. The quadratic identity (obtained by expressing the fact that (1) is a special linear complex) is

$$R \equiv (l'l')r_1^2 + (l''l'')r_2^2 + (l'''l''')r_3^2 + 2(l'l'')r_1 r_2 + 2(l''l''')r_2 r_3 + 2(l'''l')r_3 r_1 = 0,$$

where $(l'l'')$, for example, is the simultaneous invariant of l' and l''.

If we introduce ternary coördinates for each pencil of minimal lines or generators in either of these ways, a point may be defined by the six coördinates of the two minimal lines passing through it,

$$(2) \qquad r_1 : r_2 : r_3, \quad s_1 : s_2 : s_3,$$

connected by two quadratic identities

$$(3) \qquad R = 0, \quad S = 0.$$

The group G_6 then consists of those linear transformations of the r's and s's, which preserve the identities (3). See § 10.

A curve whose partial orders are $2m$, $2n$ (so that the corresponding binary form is $a_\lambda^{2m} a_\mu^{2n}$) is represented in the present coördinates by an equation of the form:

$$(4) \qquad f'(\overbrace{r_1, r_2, r_3}^{m}; \overbrace{s_1, s_2, s_3}^{n}) = 0.$$

From (3) the same curve is represented by

$$f' + MR + NS = 0, \tag{5}$$

where M, N are any double ternary forms of orders $(m-2, n)$ and $(m, n-2)$, respectively; this lack of correspondence between the curves and the forms may be removed, as in § 3, by the introduction of apolarity:

Of the system of forms $f' + MR + NS$, one and only one, (f), is identically apolar to R and S.

Combining this result with that obtained above for G_6 we have:

The inversion theory of a curve $C_{2m,\,2n}$ is equivalent to the study of the forms:

$$f = b_r^m \beta_s^n, \quad R = p_r^2, \quad S = q_s^2, \tag{6}$$

where f is identically apolar to R and S, and the ternary variables r, s undergo independent linear transformations.

If r and s are considered as homogeneous point coördinates in distinct planes, $R=0$ represents a conic in the first plane, $S=0$ a conic in the second plane, and (4) a reciprocity of orders (m, n) between the two planes; i. e., to each point r of the first plane corresponds, by means of $f'=0$, a curve K_r of order n in the second plane, and to each point s of the second plane, a curve K_s of order m in the first plane. Between the points of the two conics, the reciprocity $f'=0$, establishes a $(2m, 2n)$ correspondence as follows: to each point of the conic R corresponds, on the conic S, the $2n$ points cut out by the curve K_r, etc. The same correspondence being established by all the forms (4), the theorems above may be restated thus:

Of all the reciprocities (m, n) which establish the same $(2m, 2n)$ correspondence between the points of two conics R, S, one and only one possesses the property that all the curves K_r are apolar to S, and all the curves K_s apolar to R.

The inversion geometry of a $C_{2m,\,2n}$ has for an equivalent the study of the reciprocity (m, n) between two planes, in connection with a conic in each plane—the reciprocity being apolar to each of the conics, and the planes undergoing independent projective transformations.

§ 37. *Application to the cyclic.*—The above method is limited to the case where the partial orders of the curve are even, so that the simplest application is to the cyclics. The forms (6) § 36 are in this case

$$f = \sum b_{ik} r_i s_k = b_r \beta_s, \quad R = p_r^2, \quad S = q_s^2, \tag{1}$$

and no reduction to apolar form is necessary. The last theorem of the preceding article takes the form:

The theory of a correlation between two planes, with reference to two fundamental conics in those planes, gives a representation of the inversion geometry of the cyclic.

Let u_i, v_i denote line coördinates in the two planes, and B_{ik} the minor of b_{ik} in $|b_{ik}|$; then the correlation

$$f = \sum b_{ik} r_i s_k = 0$$

may also be written in any of the forms:

$$(2) \qquad u_i = \sum_k b_{ik} s_k, \quad v_i = \sum_k b_{ki} r_k, \quad s_i = \sum_k B_{ki} u_k, \quad r_i = \sum_k B_{ik} v_k,$$

and the two points of S, which correspond to a point r on R, are cut out by the line v given by the second of these representations.

The correlation transforms R into another conic S', and S into R', so that in each plane we have a pair of conics by whose aid the fundamental points of the $(2, 2)$ correspondence are determined as follows:

The "branch points" of R (i. e., *the points for which the corresponding s points coincide) are* $L_1L_2L_3L_4$, *the intersections of* R *and* R'; *and similarly the branch points* $M_1M_2M_3M_4$ *of* S *are the intersections of* S *and* S'. *The "double points" of* R (i. e., *the points corresponding to the branch points of* S), $L_1'L_2'L_3'L_4'$, *are the points of contact on* R *of the tangents common to* R *and* R'; *the double points of* S, $M_1'M_2'M_3'M_4'$, *are obtained similarly from the tangents common to* S *and* S'.

The four sets of points thus obtained represent the biquadratics D_1, D_2, E_1, E_2 of § 31; and the above construction furnishes a simple synthetic proof of CAYLEY's theorem (§ 31): the pair of conics S', S being correlative to both R, R' and S, S', it follows that the pairs R, R' and S, S' are homographic, and therefore the homologous sets of points $L_1L_2L_3L_4$, $M_1M_2M_3M_4$ (representing the discriminants D_1D_2) have equal anharmonic ratios on the conics R and S.

The ternary forms of the concomitants of the cyclic are most conveniently obtained from a normal form of the system (1). The binary normal form [C] § 28, is

$$(3) \qquad f = k(\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2) + l(\lambda_1^2\mu_2^2 + \lambda_2^2\eta_1^2) + 4m\lambda_1\lambda_2\mu_1\mu_2.$$

Putting

$$r_1' = \lambda_1^2, \; r_2' = 2\lambda_1\lambda_2, \; r_3' = \lambda_2^2; \quad s_1' = \mu_1^2, \; s_2' \, 2\mu_1\mu_2, \; s_3' = \mu_2^2;$$

so that the identities are

$$R = r_2'^2 - 4r_1'r_3' = 0, \quad S = s_2'^2 - 4s_1's_3',$$

equation (3) becomes

$$f = k(r_1's_1' + r_3's_3') + l(r_1's_3' + r_3's_1') + mr_2's_2';$$

and substituting

$$2ir_1' = r_1 + ir_3, \quad r_2' = r_2, \quad 2ir_3' = r_1 - ir_3, \quad \text{etc.},$$

we have the required ternary normal form:

$$[D] \qquad \begin{aligned} f &= b_1 r_1 s_1 + b_2 r_2 s_2 + b_3 r_3 s_3 \,, \\ R &= r_1^2 + r_2^2 + r_3^2 \,, \quad S = s_1^2 + s_2^2 + s_3^2 \,, \end{aligned}$$

where

$$(3) \qquad 2b_1 = -k - l \,, \quad b_2 = m \,, \quad 2b_3 = k - l \,.$$

From the concomitants of $[C]$ given in §§ 28, 31, and the transformation formulæ above, the concomitants of $[D]$ are found to be

$$I = -2\sum b_i^2 \,, \quad J = -8b_1 b_2 b_3 \,, \quad K = \sum b_i^4 - 2\sum b_i^2 b_k^2$$

(or instead of K we may consider $K' = I^2 - 4K = 16\sum b_i^2 b_k^2$);

$$f = \sum b_i r_i s_i \,, \quad g = -2\sum b_k b_l r_i s_i \,, \quad h = \sum b_i (b_i^2 - b_k^2 - b_l^2) r_i s_i \,;$$

$$D_1 = \sum b_i^2 r_i^2 \,, \quad E_1 = 4\sum b_k^2 b_l^2 r_i^2 \,, \quad S_1 = (b_1^2 - b_2^2)(b_2^2 - b_3^2)(b_3^2 - b_1^2) r_1 r_2 r_3 \,;$$

$$D_2 = \sum b_i^2 s_i^2 \,, \quad \text{etc.}$$

The conics R, R', S, S' are covariants of $[D]$; their concomitants are therefore also concomitants of $[D]$, and expressible in terms of those just written. It is sufficient to consider one pair:

$$(4) \qquad R = \sum r_i^2 \,, \quad R' = \sum b_i^2 r_i^2 \,.$$

The invariants of these conics are

$$\Delta = 1 \,, \quad \theta = \sum b_i^2 \,, \quad \theta' = \sum b_i^2 b_k^2 \,, \quad \Delta' = b_1^2 b_2^2 b_3^2 \,;$$

and the covariants are

$$R \,, \quad R' \,, \quad R'' = \sum b_k^2 b_l^2 r_i^2 \,, \quad R''' = (b_1^2 - b_2^2)(b_2^2 - b_3^2)(b_3^2 - b_1^2) r_1 r_2 r_3 \,.$$

The expressions in terms of the fundamental forms are then,

$$(5) \quad \theta = -\tfrac{1}{2} I \,, \quad \theta' = \tfrac{1}{16} K' \,, \quad \Delta' = \tfrac{1}{64} J^2 \,, \quad R' = D_1 \,, \quad R'' = \tfrac{1}{4} E_1 \,, \quad R''' = S \,.$$

From these relations we may obtain interpretations for the invariants of the cyclic of a different character from those given in §33.

If $I = 0$, then $\theta = 0$ from (5), i. e., R' is harmonically circumscribed as to R; combining this with the construction of the double points given above, and LINDEMANN'S interpretation of the Hessian, we see that †

$I = 0$ is the condition that the double points shall form the Hessian quadruple of the branch points.

* SALMON-FIEDLER, *Kegelschnitte*, Chapter 19.

† LINDEMANN, Bulletin Société Mathématique de France, vol. 5, p. 119, 1876.

Similarly, since $\theta' = 0$ if $K' = 0$, it follows that

$K' = 0$ is the condition that the branch points shall form the Hessian quadruple of the double points.

Finally $\Delta = 0$ if $J = 0$, so that the conic R' degenerates into a pair of lines, and the conics R, R' have only two common tangents, and therefore

If $J' = 0$, the four double points reduce to two; and the pairs of points on R which correspond to the points of S, form an involution of which the two double points are the foci.

§ 38. *A series of extensions.*—An important advance in the theory of binary forms is marked by the appearance in 1866 of HESSE'S "Uebertragunsprincip." * This principle, together with the notion of apolarity, has been the source of a series of methods whose best presentation in invariant form has probably been given by FRANZ MEYER in his book, *Apolarität und rationale Curven* (1883). † The basis of these methods may be stated: the theory of binary forms, or point groups on a line, is (roughly) equivalent to the projective theory of surfaces (i. e., $k - 1$ dimensional manifolds) in a space P_k of k dimensions, with reference to a fundamental rational curve N_k of the kth order to which the surfaces are all apolar. (The possession of an apolar N_k, of course in general specializes the surfaces considered.)

The partial extension of this principle to double binary forms or the inversion geometry of curves, which was given in § 36, may be generalized thus:

The geometry of a $C_{m,n}$ where $m = hm'$, $n = kn'$, has for an equivalent the theory of an (m', n') reciprocity,

$$(a_1r_1 + a_2r_2 + \cdots + a_{h+1}r_{h+1})^{m'}(a_1s_1 + a_2s_2 + \cdots + a_{k+1}s_{k+1})^{n'} = 0,$$

between two spaces, P_h, P_k, with reference to fundamental curves N_h, N_k in those spaces—the reciprocity being identically apolar to each of the curves.

This principle gives as many methods of treating a $C_{m,n}$ as there are divisors h, k of m, n. Two choices which are always possible are 1°, $h = 1$, $k = 1$; 2°, $h = m$, $k = n$. In 1° the spaces P_h, P_k are one-dimensional, and the curves N_h, N_k need not be considered, since they coincide with the spaces; the above principle reduces them to a statement of the double binary method, the equivalent of the geometry of a $C_{m,n}$ being the theory of the double binary form $a_\lambda^m \alpha_\mu^n$. In 2°, $m' = 1$, $n' = 1$, the reciprocity is bilinear, and the condition of apolarity is superfluous: the principle then gives as a representation of the geom-

* Crelle, 1866.

† Also in four memoirs, Mathematiche Annalen, vol. 21, 1883, especially, *Ein neues Uebertragunsprincip für binäre Formen deren Ordnungszahl ein nicht prime ist.* Cf. LINDEMANN, *Ueber die Darstellung binärer Formen und ihrer Covarienten durch geometrishe Gebilde im Raume,* Mathematische Annalen, vol. 23, pp. 111–142, 1884.

etry of a $C_{m,n}$, the projective theory of a linear reciprocity between spaces of dimensionalities h, k, with reference to fixed rational curves of orders h, k in those spaces.

The characteristic feature of the methods obtained in this chapter (including as a special case the double binary) is that they are concerned with two distinct spaces or with double forms, thus differing essentially from the quaternary method. The connections between the double binary and quaternary methods, which were considered in chapter IV, admit of generalization to double ternary and higher methods; and the same is true of most of the results in chapter I.

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